

Applications of Representation Theory

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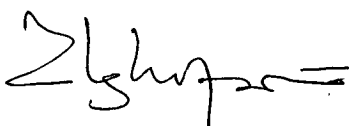
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Abdul R. Alghofari

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Abstract

This thesis presents two applications of representation theory of locally compact groups. The first is concerned with random walks, the second with Mackey's Intertwining Number Theorem.

Firstly, we consider the random walk on a collection of chambers bounded by hyperplanes in a given subspace E of \mathbf{R}^{n+1} . Initially, a particular transition probability is used in the first part of this analysis, and the identification of the collection of chambers with a reflection group provides necessary tools for obtaining a criterion for the recurrence of that walk. Next, the techniques of representation theory are used to deal with the generalization of the random walk when transition probability is considered to be a general probability measure on the group concerned.

Secondly, Mackey's Intertwining Number Theorem for one dimensional representations of open and closed subgroups of a given locally compact group G is generalized. A similar result to Mackey's is obtained in the case where the representations are finite dimensional. The recent developments in the theory of A_p^q spaces (in which such spaces are recognized as preduals of spaces of intertwining operators of induced representations) are being simplified under the condition that the subgroups are open and closed. These results, together with the fact that the space of intertwining operators between two representations can be identified with the dual of the G -tensor product of the corresponding representation spaces (endowed with the greatest cross-norm) are used to carry out the analysis.

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Chapter 1

Introduction

An interesting question concerning random walks on simple lattices was answered by mathematicians many years ago. In 1921, Pólya discovered a very beautiful application of several-dimensional Fourier series to random walks (see [4]). He thought of a particle moving on the n -dimensional lattice \mathbb{Z}^n according to the following rule. The particle starts at time 0 at the origin and moves at time $m \geq 1$ by a unit step to one of $2n$ neighboring lattice points with probability $\frac{1}{2n}$ each. The problem is to compute the probability of the event that, after m steps, the particle arrives at a fixed lattice point and to study the behavior of the particle as m tends to infinity.

Using Fourier analysis, Pólya proved that for $n \leq 2$ the ultimate behavior of the walks is recurrent, while for $n \geq 3$, it is transitory: the particle ultimately stops visiting the ball $|z| < R$, for any $R < \infty$.

Pólya's idea is to think of the probability of the particle arriving at a fixed lattice point after m steps as the corresponding Fourier coefficient of a square summable function on the standard n -dimensional Torus. Using the independence of the individual steps, this function can be simplified as a function depending only on x , m and n . Finally, the expected number of times the particle visits the origin is just the sum of the probability arriving at the origin over all

nonnegative integers.

Formally, suppose $P(s_m = k)$ denote the probability of the particle arriving at the lattice point k after m steps. Let f be a square summable function on the standard n -dimensional Torus $T^n = \{(x_1, x_2, \dots, x_n) : 0 \leq x_i < 1, 1 \leq i \leq n\}$. Then

$$f(x) = \sum_{k \in \mathbb{Z}} P(s_m = k) e^{2\pi i \langle k, x \rangle}.$$

Using the independence of the individual steps e we have

$$\begin{aligned} f(x) &= \sum_{e_1} \dots \sum_{e_n} (2n)^{-m} e^{2\pi i \langle e_1, x \rangle} \dots e^{2\pi i \langle e_m, x \rangle}, \\ &= \left[(2n)^{-1} \sum_{e_1} e^{2\pi i \langle e_1, x \rangle} \right]^m, \\ &= \left[\frac{\cos 2\pi x_1 + \dots + \cos 2\pi x_m}{n} \right]^m, \\ &= [f_n(x)]^m. \end{aligned}$$

Hence

$$P(s_m = k) = \hat{f}(k) = (f_n^m)^\wedge(k) = \int_{T^n} f_n^m(x) e^{-2\pi i \langle k, x \rangle} dx,$$

and in particular, the probability of the particle arrive at the origin is given by

$$P(s_m = 0) = \int_{T^n} f_n^m(x) dx.$$

Then the expected number of times the particle visits the origin can be derived and is given by

$$\sum_{m=0}^{\infty} P(s_m = 0) = \int_{T^n} (1 - f_n(x))^{-1} dx,$$

from which the Pólya's result is obtained. A succinct elementary account can be found in [9].

In this thesis, as our first application of theory of representations, we consider a random walk on a collection of chambers bounded by hyperplanes in a given subspace E of \mathbf{R}^{n+1} . The aim is to study the ultimate behaviour of the walk under a given transition probability.

Chapter 2 deals with the constructions of the collection of chambers for the walk. We consider the subspace E of \mathbf{R}^{n+1} given by

$$E = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\} \simeq \mathbf{R}^n.$$

The set of vectors, which are called the system of roots, given by

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n\} \cup \{e_1 - e_{n+1}\},$$

spans the subspace E , where the vectors e_i , $1 \leq i \leq n+1$ form a canonical basis in \mathbf{R}^{n+1} . We define E -hyperplanes $P_{r,k}$, r in Δ , k in \mathbf{Z} as

$$P_{r,k} = \{x \in E : \langle x, r \rangle = k\}.$$

Let L be the dual lattice in E generated by the system of root Δ ;

$$L = \{x \in E : x = \sum_{i=1}^n m_i r_i, m_i \in \mathbf{Z}\}.$$

Accordingly we define the lattice L^* of L as

$$L^* = \{x \in E : \langle x, r \rangle \in \mathbf{Z}, \text{ for all } r \in \Delta\}.$$

In this chapter, we also give a brief review of general results concerning the geometry of the reflection group required to carry out our analysis. The group in concern is the infinite group A generated by the reflections with respect to the hyperplanes $P_{r,k}$. The main point in this chapter is the fact that the group A is a semidirect product of the normal abelian subgroup T generated by the set of all translation along r , r in Δ , and the subgroup S generated by the set of reflections with respect to $P_{r,0}$, r in Δ .

Chapter 3 deals with a particular random walk by considering a particle moving on the collection of chambers in E bounded by the hyperplanes $P_{r,k}$, $r \in \Delta$, $k \in \mathbf{Z}$, according to the following rule. The particle starts at time zero at the chamber C_0 bounded by the hyperplanes $P_{r_i,0}$, $i = 1, \dots, n$, and $P_{r_0,1}$, and moves

at times $m \geq 1$ by a step across a wall into a neighboring chamber with the transition probability μ given by

$$\mu(C, C') = \begin{cases} \frac{1}{n+1}, & \text{if } C' \text{ is one of the } n+1 \text{ adjacent chambers of } C, \\ 0, & \text{otherwise.} \end{cases}$$

The steps are statistically independent of the preceding steps. The fact that this walk can be considered as a random walk in the infinite group A enables us to investigate the recurrence of the walk using the properties of the Fourier transform of functions on A . Here we obtain the result that the ultimate behavior of the walk is recurrent if $n = 2$ and transitory if $n > 2$.

In Chapter 4, we generalize our result using general probability measure μ with assumption that the support of μ generates the group A . Using theory of representations, particularly the method of “little groups” introduced by Wigner and Mackey, we construct all irreducible unitary representations of A and analyze the ultimate behavior of the walk. Here we are led to the conclusion that the random walk is recurrent or transitory according as

$$\lim_{\theta \uparrow 1} \int_{E/L^*} \text{Tr} \left\{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \right\} dx$$

is infinite or finite respectively.

Chapter 5 is devoted to our second application of representation theory of locally compact groups, namely, a generalization of Mackey’s Intertwining Number Theorem for one dimensional representations (see [16]). To explain this result, let G be a locally compact group, H and K be open and closed subgroups of G . Let π and γ be one dimensional unitary representations of H and K respectively. Mackey’s result suggest that the intertwining number (see page 68) of the two induced representations U^π and U^γ (see page 69) of G can be expressed as a sum of intertwining numbers of representations π^x and γ^y of the groups $H^x \cap K^y$, x, y in G (see [16]). We prove that the result holds in the case where π and γ are finite dimensional representations of open and closed subgroups H and K .

To achieve this we use the results in the theory of A_p^q spaces (see [2], [23]), especially the fact that the space of intertwining operators of induced representations can be recognized as the predual of a corresponding A_p^q space.

Notations

In this thesis we use the following notations. The bold letters $\mathbf{Z}, \mathbf{N}, \mathbf{R}, \mathbf{C}$ denote integers, positive integers, real and complex numbers respectively. As usual, for any positive integer n , \mathbf{R}^n denotes Euclidean n -space, where for every $x \in \mathbf{R}^n$ we may write

$$x = (x_1, x_2, \dots, x_n),$$

for some $x_1, \dots, x_n \in \mathbf{R}$. For any pair $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ defines an inner product on \mathbf{R}^n .

Let S be a set. The notation \mathbf{C}^S denotes the set of all functions defined on S with values in \mathbf{C} . It is widely accepted in functional analysis that we write tv instead of $t(v)$ for any linear transformation t acting on v , unless there is any danger of confusion. We use the notation 1_S to denote a constant function 1 defined on S . Clearly, $1_S(s) = 1$ for all $s \in S$. We denote ε_s the indicator function at s in S :

$$\varepsilon_s(t) = \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases}$$

For any group A , e_A is the identity element of A . The set of all formal linear combination of A with complex coefficient is denoted by $\mathbf{C}[A]$.

Chapter 2

Geometry

This chapter is devoted to review some background information involving the geometry of reflection and translation groups required in this thesis. Some standard results are quoted without proof; complete proof can be found in [1] and [6]. In Section 2.1, we construct a collection of chambers in a given subspace E of $(n + 1)$ -Euclidean space \mathbf{R}^{n+1} . We discuss translations and reflections in the subspace E in Section 2.2. Section 2.3 give some important results and their elementary proofs on reflection and translation groups.

2.1 Construction of the Collection of Chambers in E

The three-dimensional simple lattice \mathbf{Z}^3 is defined as the set of all triples $v = (v_1, v_2, v_3)$ with integer entries. This can be thought of as the set of all linear combinations of $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ with integer coefficients. In other words, every $v \in \mathbf{Z}^3$ can be written in the form

$$v = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1),$$

where v_1, v_2 and v_3 are integers. Any simple lattice \mathbf{Z}^n , for any positive integer n , is defined similarly.

We consider the vector space \mathbf{R}^{n+1} over the scalar field \mathbf{R} . Let 1 denote the element $(1, \dots, 1)$ of \mathbf{R}^{n+1} , and E be the annihilator of 1 with respect to the inner product $\langle \cdot, \cdot \rangle$. Then E is a closed subspace of \mathbf{R}^{n+1} and is of dimension n . Hence we have

$$E = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\} \simeq \mathbf{R}^n.$$

Let Δ be the set of vectors (called the system of roots) given by

$$\Delta = \{e_i - e_{i+1} : 1 \leq i \leq n\} \cup \{e_1 - e_{n+1}\},$$

where $\{e_1, e_2, \dots, e_{n+1}\}$ is the canonical basis for \mathbf{R}^{n+1} . It is clear that Δ spans E . Indeed, every $x \in E$ can be written in the form

$$\begin{aligned} x &= (x_1, \dots, x_n, x_{n+1}), \\ &= (x_1, x_2, \dots, x_n, -x_1 - x_2 - \dots - x_n), \\ &= x_1 r_1 + (x_1 + x_2) r_2 + \dots + (x_1 + x_2 + \dots + x_n) r_n. \end{aligned}$$

For the rest of this thesis we use the following notations unless otherwise stated:

$$\begin{aligned} r_i &= e_i - e_{i+1}, \quad 1 \leq i \leq n, \text{ and} \\ r_0 &= e_1 - e_{n+1} = \sum_{i=1}^n r_i. \end{aligned}$$

For $n = 2$, the subspace E is given in figure (2.1).

We define the lattice L in the usual way as the set generated by the system of roots Δ given by

$$L = \{x \in E : x = \sum_{i=1}^n m_i r_i, m_i \in \mathbf{Z}, r_i \in \Delta\}. \quad (2.1)$$

Here each lattice point v in L is a linear combination of vectors r_1, \dots, r_n with integer coefficients. Accordingly, we define the dual lattice L^* of L as

$$L^* = \{x \in E : \langle x, v \rangle \in \mathbf{Z}, \text{ for all } v \in L\}.$$

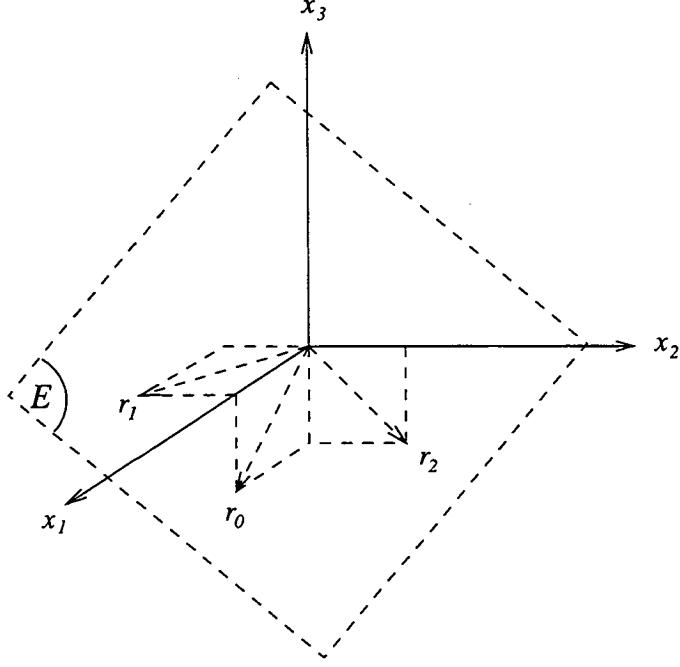


Figure 2.1: Subspace E for $n = 2$

Since the lattice L is generated by the system of roots Δ , the dual Lattice L^* is given by

$$L^* = \{x \in E : \langle x, r \rangle \in \mathbf{Z}, \text{ for all } r \in \Delta\}. \quad (2.2)$$

We consider the E -hyperplanes $P_{r,k}$ given by

$$P_{r,k} = \{x \in E : \langle r, x \rangle = k, r \in \Delta, k \in \mathbf{Z}\}.$$

The subspace E is cut out into regions or chambers by $P_{r,k}$. For $n = 2$, let x be an element of $P_{r_1,0}$. Then $x = (x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = 0$ and $\langle x, r_1 \rangle = 0$. But $\langle x, r_1 \rangle = 0$ if and only if $x_1 - x_2 = 0$. Hence $P_{r_1,0}$ is the intersecting line between two planes

$$x_1 + x_2 + x_3 = 0$$

and

$$x_1 - x_2 = 0.$$

In general, for $n = 2$, $P_{r_i,k}$ is the intersecting line between two planes

$$x_1 + x_2 + x_3 = 0$$

and either

$$x_i - x_{i+1} - k = 0,$$

for $i = 1, 2$, or

$$x_0 - x_3 - k = 0,$$

for $i = 0$.

For $n = 2$, the E -hyperplanes $P_{r_i,k}$, $i = 0, 1, 2$, $k = -1, 0, 1$, are depicted in figure (2.2).

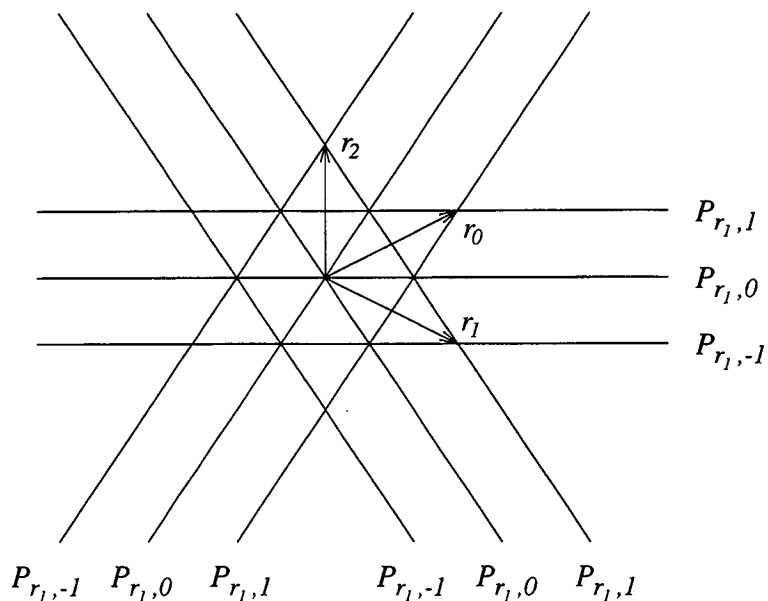


Figure 2.2: The E -hyperplanes $P_{r_i,k}$, $i = 0, 1, 2$, $k = -1, 0, 1$, for $n = 2$

2.2 Reflections and Translations in E

A reflection s with respect to a hyperplane H is a transformation that carries each vector to its mirror image with respect to the hyperplane H . We recall that

the reflection $s_{r,k}$ with respect to E -hyperplane $P_{r,k}$, for all r in Δ and k in \mathbf{Z} , is given by

$$s_{r,k}(x) = x - 2 \frac{(\langle r, x \rangle - k)r}{\langle r, r \rangle},$$

for all x in E (see [6]). Since for each $r \in \Delta$,

$$\begin{aligned} \langle r, r \rangle &= \langle e_i - e_{i+1}, e_i - e_{i+1} \rangle, \\ &= \langle e_i, e_i \rangle - \langle e_i, e_{i+1} \rangle - \langle e_{i+1}, e_i \rangle + \langle e_{i+1}, e_{i+1} \rangle = 2, \end{aligned}$$

it follows that for all r in Δ , $x \in E$, we have

$$s_{r,k}(x) = x - (\langle r, x \rangle - k)r, \tag{2.3}$$

From the formula of the reflection $s_{r,0}$, r in Δ , and our definition of the lattices L and its dual L^* , we derive the following lemma.

Lemma 2.2.1 *Let L, L^* be as in equations (2.1), (2.2) respectively and $s_{r,0}$ be the reflections with respect to the hyperplane $P_{r,0}$, r in Δ . Then, for all r in Δ , the following statements hold.*

1. $s_{r,0}$ is a linear transformation preserving the inner product $\langle \cdot, \cdot \rangle$. In particular, $s_{r,0}$ is an orthogonal transformation.
2. $s_{r,0}L = L$.
3. $s_{r,0}L^* = L^*$.

Proof. 1. Since the second statement is a direct consequence of the first, we only need to show the first. We observe, for all $x \in E$, for all r in Δ ,

$$s_{r,0}(\alpha x) = \alpha x - \langle r, \alpha x \rangle r = \alpha(x - \langle r, x \rangle r) = \alpha s_r(x),$$

for all real numbers α . Also, we have

$$\begin{aligned} s_{r,0}(x+y) &= x+y - \langle r, x+y \rangle r, \\ &= x - \langle r, x \rangle r + y - \langle r, y \rangle r, \\ &= s_{r,0}(x) + s_{r,0}(y), \end{aligned}$$

for all $x, y \in E$, r in Δ . Hence $s_{r,0}$ is a linear transformation for each r in Δ . We check, for all $x, y \in E$,

$$\begin{aligned} \langle s_{r,0}(x), s_{r,0}(y) \rangle &= \langle x - \langle r, x \rangle r, y - \langle r, y \rangle r \rangle, \\ &= \langle x, y \rangle - \langle r, y \rangle \langle x, r \rangle - \langle r, x \rangle \langle r, y \rangle + \langle r, x \rangle \langle r, y \rangle \langle r, r \rangle, \\ &= \langle x, y \rangle, \end{aligned}$$

for all r in Δ . The reflection $s_{r,0}$ therefore preserves the inner product $\langle \cdot, \cdot \rangle$ for all r in Δ .

2. Let $v = m_1 r_1 + \dots + m_n r_n$ be an element of L . Then, for $i = 0, \dots, n$, by linearity and definition of $s_{r_i,0}$,

$$\begin{aligned} s_{r_i,0}(v) &= s_{r_i,0}(m_1 r_1 + \dots + m_n r_n), \\ &= m_1 s_{r_i,0}(r_1) + \dots + m_n s_{r_i,0}(r_n), \\ &= m_1 (r_1 - \langle r_i, r_1 \rangle r_i) + \dots + m_n (r_n - \langle r_i, r_n \rangle r_i), \\ &= m_1 r_1 + \dots + m_{i-1} r_{i-1} + (m_i - \sum_{j=1}^n \langle r_i, r_j \rangle) r_i + m_{i+1} r_{i+1} + \dots + m_n r_n. \end{aligned}$$

Since $\langle r_i, r_j \rangle$ is an integer, for all $i, j = 1, \dots, n$, we conclude that $s_{r_i,0}(v)$ is an element of L for all $i = 0, \dots, n$. Hence we have $s_{r,0}L \subseteq L$, for all r in Δ . Conversely, for any $i = 0, 1, \dots, n$ and v in L , the equation

$$v = s_{r_i,0}(v')$$

has vector solution v' in L , indeed

$$v' = (s_{r_i,0} s_{r_i,0})(v') = s_{r_i,0}(v).$$

Hence v is an element of $s_{r,i,0}L$, for all $i = 0, \dots, n$. In other words, $L \subseteq s_{r,0}L$, for all r in Δ . Part (2) therefore follows.

3. Let $x \in L^*$. Then $\langle x, r_i \rangle \in \mathbf{Z}$, $i = 0, \dots, n$. We observe

$$\begin{aligned} \langle s_{r,0}(x), r_i \rangle &= \langle x - \langle r, x \rangle, r_i \rangle, \\ &= \langle x, r_i \rangle - \langle r, x \rangle \langle r, r_i \rangle \in \mathbf{Z}. \end{aligned}$$

for all $i = 0, \dots, n$. Hence $s_{r,0}(x) \in L^*$ for all $x \in L^*$, r in Δ . Thus, we have $s_{r,0}L^* \subseteq L^*$. Conversely, given $x \in L^*$, it follows that $s_{r,0}(x)$ is an element of $s_{r,0}L^*$, for all r in Δ . But, since $s_{r,0}L^* \subseteq L^*$, we have $s_{r,0}(x) \in L^*$. Hence $s_{r,0}(x) = y$ for some $y \in L^*$, from which it follows that $x = s_{r,0}(y)$ for some $y \in L^*$. Hence x is an element of $s_{r,0}L^*$. Thus we have $L^* \subseteq s_{r,0}L^*$, for all r in Δ . We conclude therefore that $s_{r,0}L^* = L^*$, for all r in Δ . □

The translation t_v by a vector v in L is given by

$$t_v(x) = x + v,$$

for all x in E . Since for all $x, y \in E$, $i = 0, \dots, n$, $t_r(x) - t_r(y) = x - y$, the translation t_r by a vector r preserves vector subtraction for all r in Δ .

Similar to Lemma 2.2.1, concerning the translations t_r , r in Δ , and the lattices L and L^* , we have the following lemma.

Lemma 2.2.2 *Let t_r be a translation by a vector r , r in Δ , L and L^* be as in Lemma 2.2.1. Then we have*

1. $t_r L = L$, and
2. $t_r L^* = L^*$.

Proof. 1. Let $x = m_1r_1 + \dots + m_nr_n$ be an element of L . Then $t_{r_i}(x) = m_1r_1 + \dots + (m_i + 1)r_i + \dots + m_nr_n$, $i = 0, \dots, n$. Hence $t_{r_i}(x)$ is an element of L , for all $i = 0, \dots, n$. Thus we have $t_rL \subseteq L$, for all r in Δ . Conversely, $x - r_i$ remains an element of L . Hence $x = t_{r_i}(x - r_i)$ is an element of $t_{r_i}L$, for all $i = 0, \dots, n$. We thus have $L \subseteq t_rL$, for all r in Δ . Therefore, we conclude $t_rL = L$, for all r in Δ .

2. Let $x \in L^*$. Then for all $i = 0, \dots, n$,

$$\langle t_r(x), r_i \rangle = \langle x + r, r_i \rangle = \langle x, r_i \rangle \langle r, r_i \rangle \in \mathbf{Z}.$$

Hence $t_r(x)$ is an element of L^* for all $x \in L^*$, for all r in Δ . Thus, we have $t_rL^* \subseteq L^*$, for all r in Δ . Conversely, let x be an element of L^* and r in Δ . Obviously, $x - r$ is in L^* . Hence $x = t_r(x - r)$ is an element of t_rL^* . Thus we have $L^* \subseteq t_rL^*$, for all r in Δ . Therefore (2) follows. □

We have the following useful lemma concerning the relation between the reflection $s_{r,k}$ and the translation t_r , r in Δ , k in \mathbf{Z} .

Lemma 2.2.3 *Let $s_{r,k}$ be as in equations (2.3) and t_r be a translation by r , r in Δ , $k \in \mathbf{Z}$. Then*

$$1. \ s_{r,k} = t_{kr}s_{r,0}, \text{ and}$$

$$2. \ t_r = s_{r,1}s_{r,0}.$$

Proof. 1. For $x \in E$, we have

$$\begin{aligned} (t_{kr}s_{r,0})(x) &= t_{kr}(x - \langle r, x \rangle r), \\ &= x - \langle r, x \rangle r + kr, \\ &= x - (\langle x, r \rangle - k)r, \\ &= s_{r,k}(x), \end{aligned}$$

for all r in Δ . Part (1) thus immediately follows.

2. For every $x \in E$, we have

$$\begin{aligned}
 (s_{r,1}s_{r,0})(x) &= s_{r,1}(x - \langle r, x \rangle r), \\
 &= x - \langle r, x \rangle r - (\langle r, x - \langle r, x \rangle r \rangle - 1)r, \\
 &= x - \langle r, x \rangle r - \langle r, x \rangle r \langle r, x \rangle \langle r, r \rangle r + r, \\
 &= x + r = t_r(x),
 \end{aligned}$$

for all r in Δ , since $\langle r, r \rangle = 2$. Hence we obtain $t_r = s_{r,1}s_{r,0}$.

□

2.3 Translation and Reflection Groups

Lemma 2.2.1 and Lemma 2.2.2 give us a motivation to construct groups generated by the reflections and translations we have discussed in those lemmas. This motivation is formulated in the following two lemmas.

Lemma 2.3.1 *Let S be the group generated by $s_{r,0}$, T be the group generated by t_r , r in Δ , L be as in Lemma 2.2.1. Then the following statements hold.*

1. $T = \{t_v : v \in L\} \simeq L$.
2. $st_v = t_{s(v)}s$, for all $s \in S$, $v \in L$.
3. $sL = L$, for all $s \in S$.
4. $T \cap S = \{e_A\}$, where e is the identity element.

Proof. 1. Let v be an element of L . Then $v = \sum_{i=1}^n m_i r_i$, $m_i \in \mathbb{Z}$. Hence we have

$$t_v = t_{\sum_{i=1}^n m_i r_i},$$

$$\begin{aligned}
&= t_{m_1 r_1} t_{m_2 r_2} \dots t_{m_n r_n}, \\
&= (t_{r_1})^{m_1} (t_{r_2})^{m_2} \dots (t_{r_n})^{m_n}.
\end{aligned}$$

Hence t_v is an element of T . Thus we have $\{t_v : v \in L\} \subseteq T$. Conversely, let t be an element of T . Since T is obviously a commutative group, we may write

$$\begin{aligned}
t &= (t_{r_0})^{p_0} \dots (t_{r_n})^{p_n}, \\
&= t_{p_0 r_0 + \dots + p_n r_n}, \\
&= t_{(p_0 + p_1) r_1 + \dots + (p_0 + p_n) r_n},
\end{aligned}$$

for some $p_i \in \mathbb{Z}$, $i = 0, \dots, n$. Hence t is an element of $\{t_v : v \in L\}$. Thus $T \subseteq \{t_v : v \in L\}$. Finally, let us define a function $\pi : L \longrightarrow T$ given by

$$\pi(v) = t_v,$$

for all $v \in L$. Then π is clearly a one to one function from L onto T . Moreover, π is a homomorphism, since for all v_1, v_2 in L , we have

$$\pi(v_1 + v_2) = t_{v_1 + v_2} = t_{v_1} t_{v_2} = \pi(v_1) \pi(v_2).$$

Hence π is an isomorphism from L onto $\{t_v : v \in L\}$.

2. Let s be an element of S and v be an element of L . Then

$$(st_v)(x) = s(t_v(x)) = s(x + v) = sx + sv = t_{sv}(sx) = (t_{sv}s)(x).$$

Thus we obtain for all s in S , v in L ,

$$st_v = t_{sv}s.$$

3. Since the group S is generated by $s_{r,0}$, r in Δ , part (3) follows directly from Lemma 2.2.1 (2).

4. Let u be any element of $T \cap S$. Then $u = t_v$ for some v in L and $u = s$ for some s in S . Now by (2) we obtain

$$ut_v = st_v = t_{sv}s = t_{sv}t_v.$$

Since $u = t_v$, for some v in L , we find $t_v = t_{sv}$. Hence we have

$$v = sv.$$

But, since $s = t_v$, we obtain

$$v = t_v(v) = 2v.$$

Thus v must be identical to zero. This implies that t_v is the identity transformation, or in other words, $u = e_A$.

□

Having discussed the relation between the generators of the groups S and T , we close this chapter with an important lemma concerning the groups S and T which will be used in our work in Chapter 3 and 4.

Lemma 2.3.2 *Let A be the group generated by $s_{r,k}$, r in Δ S and T be as in Lemma 2.3.1. Then the following statements hold.*

1. S is a subgroup of A and T is a normal subgroup of A .
2. $A = TS$, in the sense that every element a in A can be written uniquely as

$$a = t_v s,$$

where $t_v \in T$ and $s \in S$.

3. For $s_1, s_2 \in S$, and for $t_{v_1}, t_{v_2} \in T$, $(t_{v_1} s_1)(t_{v_2} s_2) = t_v s$, where $v = v_1 + s_1(v_2)$ and $s = s_1 s_2$.

Proof. 1. The statement that S is a subgroup of A is obvious from the definitions of the groups S and A . Now from Lemma 2.2.3 (2), it implies that $t_r \in A$, for all

r in Δ . Hence we infer that T is a subgroup of A . We shall prove now that T is normal. Let $a \in A$. Then by Lemma 2.2.3 (1) there exist an integer m such that

$$\begin{aligned} a &= s_{u_1, k_1} s_{u_2, k_2} \cdots s_{u_m, k_m}, \\ &= t_{k_1 u_1} s_{u_1, 0} t_{k_2 u_2} s_{u_2, 0} \cdots t_{k_m u_m} s_{u_m, 0}, \end{aligned} \quad (2.4)$$

where k_i are integers and u_i in Δ , $i = 1, \dots, m$. By Lemma 2.3.1 (3) and the fact that L is invariant under the acts of $t_{k_i u_i}$, we conclude that $a(v)$ is an element of L for every v in L . Therefore, by the same way we proved Lemma 2.3.1 (2), we have

$$at_v = t_{a(v)}a.$$

Thus ata^{-1} is an element of T from which it follows that T is a normal subgroup of A .

2. To prove (2) it suffices to prove that $A = TS$ and $T \cap S = \{e_A\}$. By Lemma 2.3.1 (4) we have the latter. Obviously $TS \subseteq A$. We shall now show that $A \subseteq TS$. By Lemma 2.2.3 (1), every element a in A can be written as in equation (2.4). Since S and T are subgroup of A , and, by Lemma 2.3.1 (2), $TS = ST$, it follows that TS is a subgroup of A and hence a is an element of TS . Thus $A = TS$.

3. By Lemma 2.3.1 (2), we have

$$(t_{v_1} s_1)(t_{v_2} s_2) = t_{v_1}(s_1 t_{v_2})s_2 = t_{v_1} t_{s_1 v_2} s_1 s_2 = t_{v_1 + s_1(v_2)} s_1 s_2,$$

for all $s_1, s_2 \in S$ and for all $t_{v_1}, t_{v_2} \in T$.

□

Chapter 3

Random Walk

There has been much work concerning the theory of random walks on groups. The theory was first raised by Kesten (see [11]) and a number of influential and valuable papers are presented for example by Kaimanovich and Vershik (see [10]), Varopoulos (see [28]) and others. A good survey may be found in [29].

The main results in this chapter are given in Theorem 3.4.4 and Corollary 3.4.5 which can be derived directly from Varopoulos' result (see [29] page 12).

In Chapter 2, we defined hyperplanes $P_{r,k}$, r in Δ , in the subspace E of \mathbf{R}^{n+1} from which we obtained a collection of chambers. The purpose of this chapter is to study a particular random walk defined in this collection of chambers and analyze the ultimate behaviour of the walk.

First, we established the fact that the collection of chambers can be identified with the reflection group A . Then the symmetric transition probability of the walk can be regarded as a function on the group A . We use the properties of the Fourier transform of functions on the group A to simplify our formula which leads to the criterion for the recurrence.

3.1 A Random Walk with a Particular Transition Probability

Let \mathcal{C} be the collection of chambers which are region in E cut out by E -hyperplane $P_{r,k}$, r in Δ and k in \mathbf{Z} . Suppose a particle moves on the collection of chambers \mathcal{C} according to the following rule. For $n = 2$ (E is 2-dimensional subspace), the collection of chambers is depicted in figure 3.1.

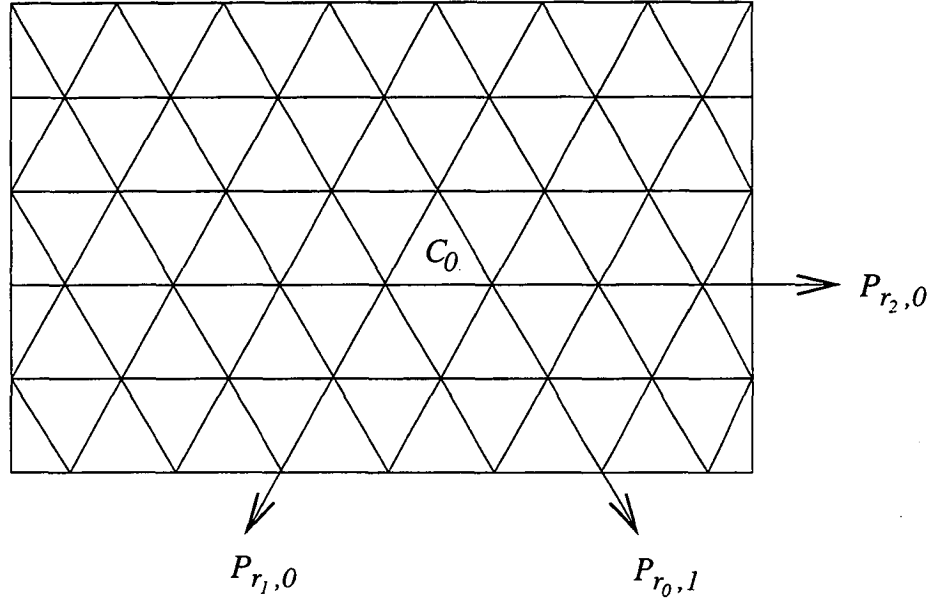


Figure 3.1: Random walks on the collection of chambers \mathcal{C}

The particle starts at the chamber C_0 and moves at times $m \geq 1$ by a unit step to an adjacent chamber, where $C_0 = \{x \in E : \langle r_i, x \rangle > 0 \text{ for } i = 1, \dots, n, \text{ and } \langle r_0, x \rangle < 1\}$ with probability $\frac{1}{n+1}$ each. The steps are statistically independent of the preceding steps.

Let μ be the transition probability on the collection of chambers \mathcal{C} . Then μ is given by

$$\mu(C, C') = \begin{cases} \frac{1}{n+1}, & \text{if } C' \text{ is one of the } n+1 \text{ adjacent chambers of } C, \\ 0, & \text{otherwise.} \end{cases}$$

It is beneficial for our purposes to consider the transition probability μ as a function on $A \times A$ instead of a function on $\mathcal{C} \times \mathcal{C}$. For this purpose we shall prove the following Lemma.

Lemma 3.1.1 *The group A permutes the collection of chambers \mathcal{C} simply transitively, in other words, for the fixed chamber C_0 , the rule $\psi : A \longrightarrow \mathcal{C}$ given by $\psi(a) = aC_0$, for all a in A , is a one-to-one mapping from the group A onto the collection of chambers \mathcal{C} .*

Proof. This is proved in detail in [1]. Here we only show that the group A permutes the chambers. This immediately follows, because

1. The transformation $s_{r,0}$ is orthogonal and the translation t_r preserves vector subtraction, for all r in Δ . In other words, under the action of a , for all a in A , the shape of every chamber is unchanged.
2. L^* is invariant under the action of $s_{r,0}$ and of t_r , for all r in Δ .

□

Because of the fact that ψ is one to one mapping from the group A onto the collection of chambers \mathcal{C} , we can identify each element in \mathcal{C} with a unique element in A . Therefore, we can write $\mu(a_1, a_2) = \mu(C_1, C_2)$, if $C_1 = a_1C_0$ and $C_2 = a_2C_0$, for a_1, a_2 in A and C_1, C_2 in \mathcal{C} . As a result, we can consider the transition probability μ as a function on $A \times A$ rather than on $\mathcal{C} \times \mathcal{C}$. For brevity we set the notation s_r to denote $s_{r,0}$, for all r in Δ , unless otherwise stated.

Lemma 3.1.2 *For every a, a_1, a_2 in A ,*

1. $\mu(aa_1, aa_2) = \mu(a_1, a_2)$.

$$\begin{aligned}
2. \quad \mu(a_1, a_2) &= \mu(e_A, a_1^{-1}a_2) \\
&= \begin{cases} \frac{1}{n+1}, & \text{if } a_1^{-1}a_2 \in \{s_{r_1}, s_{r_2}, \dots, s_{r_n}, s_{r_0,1}\}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. 1. Let $\psi(a_1) = a_1C_0 = C_1$ and $\psi(a_2) = a_2C_0 = C_2$. Then $\mu(a_1, a_2) = \mu(C_1, C_2)$. There are two possibilities; $\mu(a_1, a_2) = 0$ or $\mu(a_1, a_2) = \frac{1}{n+1}$. If $\mu(a_1, a_2) = 0$, then C_1 is not adjacent to C_2 . Suppose $\mu(aa_1, aa_2) \neq 0$. Then aC_1 is adjacent to aC_2 . But, since A acts on \mathcal{C} transitively $a^{-1}aC_1 = C_1$ is adjacent to $a^{-1}aC_2 = C_2$, which is a contradiction. Hence $\mu(aa_1, aa_2) = 0$. If $\mu(a_1, a_2) = \frac{1}{n+1}$, then C_1 and C_2 have a common boundary. Since this boundary will be mapped on the common boundary of aC_1 and aC_2 by the element a in A , we have $\mu(aC_1, aC_2) = \mu(aa_1, aa_2) = \frac{1}{n+1}$.

2. To show the first equality, by (1) we observe

$$\begin{aligned}
\mu(a_1, a_2) &= \mu(a_1^{-1}a_1, a_1^{-1}a_2), \\
&= \mu(e_A, a_1^{-1}a_2).
\end{aligned}$$

We will now show the second equality. Using the definition of μ , we have

$$\begin{aligned}
\mu(a_1, a_2) &= \mu(e_A, a_1^{-1}a_2), \\
&= \mu(C_0, a_1^{-1}a_2C_0), \\
&= \begin{cases} \frac{1}{n+1}, & \text{if } a_1^{-1}a_2 \in \{s_{r_1}, s_{r_2}, \dots, s_{r_n}, s_{r_0,1}\} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

□

From the foregoing result, the function μ can be considered as a function defined on the group A , so that we may write $\mu(a)$ instead of $\mu(e_A, a)$, for all a in A .

Lemma 3.1.3 *The probability of the particle arriving at a after m steps (starting at e_A) is given by*

$$\underbrace{(\mu \times \mu \times \dots \times \mu)}_{m \text{ times}}(a), \quad (3.1)$$

where \times denotes convolution product of functions on the group A defined by

$$(f \times g)(a) = \sum_{b \in A} f(ab^{-1})g(b), \quad (3.2)$$

$$= \sum_{b \in A} f(b)g(b^{-1}a), \quad (3.3)$$

where f, g are elements in the space $L^1(A)$ of all summable functions on A .

Proof. For simplicity we let $\mu^{\times m}(a)$ to write the expression in (3.1) above. The proof is by induction on the number of steps m . Obviously for $m = 1$ the statement is trivial. Suppose now that the statement is true for some positive integer $m = p$. Then the probability of the particle arriving at a after $p + 1$ steps is given by

$$\sum_{c \in A} \mu^{\times p}(c) \mu(c, a).$$

By equation (3.3)

$$\begin{aligned} \sum_{c \in A} \mu^{\times p}(c) \mu(c, a) &= \sum_{c \in A} \mu^{\times p}(c) \mu(c^{-1}a), \\ &= \mu^{\times(p+1)}(a). \end{aligned}$$

Hence the statement is true for $m = p + 1$. By the principle of mathematical induction the statement is true for all m in \mathbf{N} .

□

In the following two sections we explore some important results from Fourier transforms of functions on A which will be used as a technical tools to achieve the criterion for recurrence of the walk.

3.2 Fourier Transforms of Functions on A

In this section we shall mainly study about functions on the group A and derive some important results from their Fourier transform. The main object is to formulate explicitly the probability of the particle arriving at the origin after m steps. We also give the Fourier transform of convolution functions.

The indicator function ε_{e_A} of the identity element e_A of A is given by

$$\varepsilon_{e_A}(a) = \begin{cases} 1, & \text{if } a = e_A, \\ 0, & \text{otherwise.} \end{cases}$$

Then for all a in A , every function f on A can be written as

$$f(a) = (\varepsilon_{e_A} \times f)(a),$$

In fact, for all a in A ,

$$\begin{aligned} (\varepsilon_{e_A} \times f)(a) &= \sum_{b \in A} \varepsilon_{e_A}(b) f(b^{-1}a), \\ &= f(a). \end{aligned}$$

By Lemma 2.3.2 (2), each $a \in A$ can be written uniquely in the form

$$a = t_v s,$$

for some v in L and s in S . Therefore, for a in A , we may write a in the form $a = (v, s)$, for some v in L and s in S , in particular, $e_A = (0, e_S)$. As a consequence, each function f on A can be thought of as a function on $L \times S$. We write $f(v, s)$ instead of $f(a) = f(t_v s)$ for each $v \in L$ and $s \in S$.

We recall that the Fourier transform of any function $f(v, s)$ in the space $L^1(A)$ of all summable functions on A with respect to $v \in L$ is given by

$$\hat{f}(x, s) = \sum_{v \in L} f(v, s) e^{-2\pi i \langle x, v \rangle},$$

defined for all $x \in E$ and for all $s \in S$. We shall observe that $\hat{f}(x, s)$ is periodic with respect to L^* . Let x' be an arbitrary element of L^* . Then

$$\begin{aligned}\hat{f}(x + x', s) &= \sum_{v \in L} f(v, s) e^{-2\pi i \langle x + x', v \rangle}, \\ &= \sum_{v \in L} f(v, s) e^{-2\pi i \langle x, v \rangle} e^{-2\pi i \langle x', v \rangle}, \\ &= \sum_{v \in L} f(v, s) e^{-2\pi i \langle x, v \rangle}, \\ &= \hat{f}(x, s),\end{aligned}$$

since $\langle x', v \rangle$ is in \mathbf{Z} , for all v in L^* . Therefore, we can think of $\hat{f}(x, s)$ as a function on $(E/L^*) \times S$.

Lemma 3.2.1 *The probability of the particle arriving at the origin after m steps is given by*

$$\mu^{\times m}(0, e_S) = \int_{E/L^*} (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, e_S) dx, \quad (3.4)$$

where ε_{e_A} is the indicator function at e_A .

Proof. By Lemma 3.1.3, $\mu^{\times m}(0, e_S)$ represents the probability of the particle arriving at the origin after m steps. We observe now that the inverse Fourier transform of $(\mu^{\times m})^\wedge(x, s)$ is given by

$$\begin{aligned}\mu^{\times m}(v, s) &= \int_{E/L^*} (\mu^{\times m})^\wedge(x, s) e^{2\pi i \langle x, v \rangle} dx, \\ &= \int_{E/L^*} (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, s) e^{2\pi i \langle x, v \rangle} dx,\end{aligned}$$

for all (v, s) in A . In particular, substituting $(0, e_S)$ for (v, s) , the probability that the particle comes back to the origin after m steps is given by

$$\mu^{\times m}(0, e_S) = \int_{E/L^*} (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, e_S) dx.$$

□

To simplify this further, let us look at the Fourier transform of the transition probability μ and its convolution with any function f in C^A . This convolution function will be used later in Section 3.3 for defining an operator on the Hilbert space C^S .

Lemma 3.2.2 *Let f be an element in C^A . Then the Fourier transform of $\mu(v, s)$ and $(f \times \mu)(v, s)$ are given by*

$$\hat{\mu}(x, s) = \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \varepsilon_i(s) \right) + e^{-2\pi i \langle x, r_0 \rangle} \varepsilon_0(s) \right\}$$

and

$$(f \times \mu)^\wedge(x, s) = \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \hat{f}(x, s s_{r_i}) \right) + e^{2\pi i \langle x, s r_0 \rangle} \hat{f}(x, s s_{r_0}) \right\}$$

respectively, where ε_i are the indicator functions on S at s_{r_i} , $0 \leq i \leq n$.

Proof. The Fourier transform of $\mu(v, s)$ is given by

$$\hat{\mu}(x, s) = \sum_{v \in L} (\varepsilon_{e_A} \times \mu)(v, s) e^{-2\pi i \langle x, v \rangle},$$

for all x in E and s in S . But, by Lemma (3.1.2) we have

$$\mu(v, s) = \begin{cases} \frac{1}{n+1}, & \text{if } (v, s) \in \{(0, s_{r_1}), (0, s_{r_2}), \dots, (0, s_{r_n}), (r_0, s_{r_0})\}, \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

Hence

$$\hat{\mu}(x, s) = \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \varepsilon_i(s) \right) + e^{-2\pi i \langle x, r_0 \rangle} \varepsilon_0(s) \right\},$$

where ε_i , $0 \leq i \leq n$, are indicator functions on S at s_{r_i} .

Let f be a function on A . By the definition of convolution function in equation (3.2), we observe

$$\begin{aligned} (f \times \mu)^\wedge(x, s) &= \sum_{v \in L} (f \times \mu)(v, s) e^{-2\pi i \langle x, v \rangle}, \\ &= \sum_{v \in L} \sum_{(v', s') \in A} f((v, s)(v', s')^{-1}) \mu(v', s') e^{-2\pi i \langle x, v \rangle}. \end{aligned}$$

By (3.5) and the fact that $(0, s_{r_i})^{-1} = (0, s_{r_i})$ for all $i = 1, \dots, n$ and $(r_0, s_{r_0})^{-1} = (r_0, s_{r_0})$, we find

$$\begin{aligned} (f \times \mu)^\wedge(x, s) &= \sum_{v \in L} \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n f((v, s)(0, s_{r_i})) e^{-2\pi i \langle x, v \rangle} \right) + f((v, s)(r_0, s_{r_0})) e^{-2\pi i \langle x, v \rangle} \right\}. \end{aligned}$$

Using Lemma 2.3.2 (3), we have

$$(v, s)(r_0, s_{r_0}) = t_v s t_{r_0} s_{r_0} = (v + s r_0, s s_{r_0}) \text{ and } (v, s)(0, s_{r_i}) = t_v s s_{r_i} = (v, s s_{r_i}),$$

from which we obtain

$$\begin{aligned} (f \times \mu)^\wedge(x, s) &= \frac{1}{n+1} \left\{ \sum_{i=1}^n \hat{f}(x, s s_{r_i}) + \sum_{v \in L} f(v + s r_0, s s_{r_0}) e^{-2\pi i \langle x, v \rangle} \right\}, \\ &= \frac{1}{n+1} \left\{ \sum_{i=1}^n \hat{f}(x, s s_{r_i}) \right. \\ &\quad \left. + \sum_{v \in L} f(v + s r_0, s s_{r_0}) e^{-2\pi i \{ \langle x, v + s r_0 \rangle - \langle x, s r_0 \rangle \}} \right\}, \\ &= \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \hat{f}(x, s s_{r_i}) \right) + e^{2\pi i \langle x, s r_0 \rangle} \hat{f}(x, s s_{r_0}) \right\}. \end{aligned}$$

□

3.3 The Bounded Linear Operator $P(x)$

The fact that S is isomorphic to the symmetric group S_{n+1} is evident. Indeed, a simple way to see this isomorphism is to note that the reflection s_i interchanges the vector e_i and e_{i+1} and leaves all the other basis vectors fixed. Therefore, each generating reflection in S is a permutation of the basis vectors, implying that the entire group S is contained in the group of permutations of the basis vectors. Moreover, the imbedding of S into S_{n+1} is actually onto.

Let \mathbf{C}^S be the vector space of all complex valued functions on S over the scalar field \mathbf{C} . We regard $\hat{f}(x, \cdot)$ as a function in \mathbf{C}^S for all x in E/L^* . For simplicity we shall write s_i to denote s_{r_i} , for all $i = 0, \dots, n$.

For each $x \in E/L^*$, we wish to define a bounded operator $P(x)$ on the space \mathbb{C}^S so that

$$\begin{aligned} P(x)\hat{f}(x, s) &= (f \times \mu)^\wedge(x, s), \\ &= \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \hat{f}(x, ss_i) \right) + e^{2\pi i \langle x, sr_0 \rangle} \hat{f}(x, ss_0) \right\}. \end{aligned} \quad (3.6)$$

Using equation (3.6), we can accomplish this by defining the operator $P(x)$ on \mathbb{C}^S according to the formula given by

$$(P(x)\phi)(s) = \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \phi(ss_i) \right) + e^{2\pi i \langle x, sr_0 \rangle} \phi(ss_0) \right\}. \quad (3.7)$$

for all function ϕ in \mathbb{C}^S and s in S .

Observe that the set of indicator functions $\{\varepsilon_s : s \in S\}$ on S form a basis for the space \mathbb{C}^S . Any function $\psi \in \mathbb{C}^S$ can be written uniquely as $\psi = \sum_{s \in S} \psi(s)\varepsilon_s$ and therefore, can be identified as the $(n+1)!$ -tuple $(\psi(s))_{s \in S}$ with respect to the basis $\{\varepsilon_s : s \in S\}$. Accordingly, we can introduce an inner product $\langle \cdot, \cdot \rangle$ on \mathbb{C}^S defined by

$$\langle \phi, \psi \rangle = \frac{1}{|S|} \sum_{s \in S} \phi(s) \overline{\psi(s)}, \quad (3.8)$$

for all $\phi, \psi \in \mathbb{C}^S$, where $\overline{\psi(s)}$ denotes the complex conjugate of $\psi(s)$. The space \mathbb{C}^S equipped with this inner product is a complex Hilbert space. Hence the operator $P(x)$ has at least one eigenvalue.

Lemma 3.3.1 *The operator $P(x)$ is self-adjoint for each $x \in E/L^*$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined in equation (3.8).*

Proof. For simplicity let us define bounded linear operators S_i , $i = 0, \dots, n$, and $K(x)$, $x \in E/L^*$, from \mathbb{C}^S into itself by

$$(S_i\psi)(s) = \psi(ss_i), \quad (3.9)$$

and

$$(K(x)\psi)(s) = e^{2\pi i \langle x, sr_0 \rangle} (S_0\psi)(s), \quad (3.10)$$

for all ψ in \mathbf{C}^S . Then for all x in E/L^* , the operator $P(x)$ can be rewritten as

$$P(x) = \frac{1}{n+1} \left\{ K(x) + \sum_{i=1}^n S_i \right\}. \quad (3.11)$$

Therefore, to prove Lemma 3.3.1 it suffices to prove that S_i , $i = 1, \dots, n$ and $K(x)$ are self-adjoint for all x in E/L^* . To do this, let ϕ, ψ be elements in \mathbf{C}^S and i in $\{0, 1, 2, \dots, n\}$. Then

$$\begin{aligned} \langle S_i \phi, \psi \rangle &= \frac{1}{|S|} \sum_{s \in S} (S_i \phi)(s) \overline{\psi(s)}, \\ &= \frac{1}{|S|} \sum_{s \in S} \phi(ss_i) \overline{\psi(s)}. \end{aligned}$$

Changing variable from s to ss_i , we find

$$\begin{aligned} \langle S_i \phi, \psi \rangle &= \frac{1}{|S|} \sum_{s \in S} \phi(s) \overline{\psi(ss_i)}, \\ &= \langle \phi, S_i \psi \rangle. \end{aligned} \quad (3.12)$$

Similarly, if $x \in E/L^*$, then

$$\begin{aligned} \langle K(x) \phi, \psi \rangle &= \frac{1}{|S|} \sum_{s \in S} (K(x) \phi)(s) \overline{\psi(s)}, \\ &= \frac{1}{|S|} \sum_{s \in S} e^{2\pi i \langle x, sr_0 \rangle} \phi(ss_0) \overline{\psi(s)}. \end{aligned}$$

Changing variable from s to ss_0 , we obtain

$$\begin{aligned} \langle K(x) \phi, \psi \rangle &= \frac{1}{|S|} \sum_{s \in S} e^{-2\pi i \langle x, sr_0 \rangle} \phi(s) \overline{\psi(ss_0)}, \\ &= \frac{1}{|S|} \sum_{s \in S} \phi(s) \overline{e^{2\pi i \langle x, sr_0 \rangle} \psi(ss_0)}, \\ &= \langle \phi, K(x) \psi \rangle, \end{aligned} \quad (3.13)$$

since $s_0 r_0 = -r_0$ and $\overline{e^{2\pi i \langle x, sr_0 \rangle}} = e^{-2\pi i \langle x, sr_0 \rangle}$. From equations (3.12) and (3.13) we conclude that $S_i, i = 1, \dots, n$ and $K(x)$ are self-adjoint. Therefore $P(x)$ is self-adjoint for each $x \in E/L^*$. \square

Consequently, each eigenvalue of $P(x)$ is real, for all $x \in E/L^*$. Moreover, we have the following lemma.

Lemma 3.3.2 *Let λ be an eigenvalue of $P(x)$, $x \in E/L^*$. Then*

1. $|\lambda| \leq 1$.
2. $P(0)$ has 1 as a simple eigenvalue with all nonzero constant functions as the corresponding eigenvectors.
3. If $x \neq 0$, then $(I - P(x))$ is invertible.

Proof. 1. Let, λ be an eigenvalue of $P(x)$, $x \neq 0$. Then for a corresponding eigenvector φ , we have

$$\lambda\varphi = P(x)\varphi.$$

Hence for all s in S ,

$$\begin{aligned} \lambda\varphi(s) &= P(x)\varphi(s), \\ &= \frac{1}{n+1} \left\{ \left(\sum_{i=1}^n \varphi(ss_i) \right) + e^{2\pi i \langle x, sr_0 \rangle} \varphi(ss_0) \right\}. \end{aligned} \quad (3.14)$$

Since φ is nonzero function, there exists t in S such that $|\varphi(t)| \neq 0$ and hence $\sum_{s \in S} |\varphi(s)| \neq 0$. Therefore, summing the absolute value of equation (3.14) for all s in S , we obtain

$$\begin{aligned} |\lambda| \sum_{s \in S} |\varphi(s)| &= \frac{1}{n+1} \sum_{s \in S} \left| \left(\sum_{i=1}^n \varphi(ss_i) \right) + e^{2\pi i \langle x, sr_0 \rangle} \varphi(ss_0) \right|, \\ &\leq \frac{1}{n+1} \sum_{s \in S} \sum_{i=0}^n |\varphi(ss_i)|, \\ &= \frac{1}{n+1} \sum_{i=0}^n \sum_{s \in S} |\varphi(ss_i)|, \\ &= \sum_{s \in S} |\varphi(s)|, \end{aligned}$$

since $\sum_{s \in S} \varphi(ss_i) = \sum_{s \in S} \varphi(s)$, for all $i = 0, \dots, n$. Hence we find $|\lambda| \leq 1$.

2. If $x = 0$, from equation (3.7) we find

$$(P(0)\phi)(s) = \frac{1}{n+1} \sum_{i=0}^n \phi(ss_i),$$

for all $\phi \in \mathbb{C}^S$. Hence if ϕ is a constant function, then $P(0)\phi = \phi$. Therefore, $P(0)$ has an eigenvalue 1 with all nonzero constant functions as the corresponding eigenvectors. To prove that the eigenvalue 1 is a simple eigenvalue, we have to show that the eigenspace corresponding to eigenvalue 1 is of dimension one.

Let $P(0)\phi = \phi$. Then by equation (3.7) we have

$$\phi(s) = \frac{1}{n+1} \sum_{i=0}^n \phi(ss_i). \quad (3.15)$$

We observe that

$$\begin{aligned} \sum_{s \in S} \sum_{i=0}^n |\phi(s) - \phi(ss_i)|^2 &= \sum_{s \in S} \sum_{i=0}^n \left\{ |\phi(s)|^2 + |\phi(ss_i)|^2 - 2\operatorname{Re} \left(\phi(s) \overline{\phi(ss_i)} \right) \right\}, \\ &= \sum_{s \in S} \left\{ (n+1)|\phi(s)|^2 + \sum_{i=0}^n |\phi(ss_i)|^2 \right. \\ &\quad \left. - 2\operatorname{Re} \left(\phi(s) \sum_{i=0}^n \overline{\phi(ss_i)} \right) \right\}. \end{aligned} \quad (3.16)$$

By equation (3.15), we have $\sum_{i=0}^n \overline{\phi(ss_i)} = (n+1)\overline{\phi(s)}$. Hence equation (3.16) becomes

$$\begin{aligned} \sum_{s \in S} \sum_{i=0}^n |\phi(s) - \phi(ss_i)|^2 &= \sum_{s \in S} \left\{ (n+1)|\phi(s)|^2 + \sum_{i=0}^n |\phi(ss_i)|^2 - 2(n+1)|\phi(s)|^2 \right\}, \\ &= \sum_{s \in S} \left\{ \sum_{i=0}^n |\phi(ss_i)|^2 - (n+1)|\phi(s)|^2 \right\}, \\ &= \sum_{i=0}^n \sum_{s \in S} |\phi(ss_i)|^2 - (n+1) \sum_{s \in S} |\phi(s)|^2, \\ &= 0, \end{aligned}$$

since $\sum_{s \in S} |\varphi(ss_i)| = \sum_{s \in S} |\varphi(s)|$ for all $i = 1, \dots, n$. Thus $\phi(s) = \phi(ss_i)$ for all s in S and $i = 0, \dots, n$. In other words, by equation (3.9), we have

$$S_i \phi = \phi,$$

for all $i = 0, \dots, n$, from which it follows that $S_i \dots S_j \phi = \phi$, for all i, \dots, j in $\{0, \dots, n\}$. Hence we have $\phi(ss_i \dots s_j) = \phi(s)$, for all s in S for all i, \dots, j in $\{0, \dots, n\}$, and in particular, $\phi(es_i \dots s_j) = \phi(e)$ for all i, \dots, j in $\{0, \dots, n\}$. Hence it implies that ϕ is a constant function, since $\{s_0, \dots, s_n\}$ generates S . Therefore, nonzero constant functions are the only eigenvectors of $P(0)$ corresponding to eigenvalue 1. As a result, the eigenspace corresponding to eigenvalue 1 is of dimension one.

3. $(I - P(x))$ is invertible if and only if 1 is not an eigenvalue of $P(x)$. To prove part (3) we can prove its contraposition, hence it suffices to prove the implication: if 1 is an eigenvalue of $P(x)$ then $x = 0$. Before doing this, by equation (3.9), we observe

$$\|S_i \phi\|^2 = \langle S_i \phi, S_i \phi \rangle = \frac{1}{|S|} \sum_{s \in S} \phi(ss_i) \overline{\phi(ss_i)} = \|\phi\|^2,$$

for all ϕ in \mathbb{C}^S , for all $i = 0, \dots, n$. Hence the norm $\|S_i\|$ of S_i is given by

$$\|S_i\| = \sup_{\substack{\|\phi\|=1 \\ \phi \in \mathbb{C}^S}} \|S_i \phi\| = 1, \quad (3.17)$$

for all $i = 0, \dots, n$. Also, by equation (3.10),

$$\begin{aligned} \|K(x)\phi\|^2 &= \langle K(x)\phi, K(x)\phi \rangle, \\ &= \frac{1}{|S|} \sum_{s \in S} e^{2\pi i \langle x, sr_0 \rangle} \phi(ss_0) \overline{e^{2\pi i \langle x, sr_0 \rangle} \phi(ss_0)}, \\ &= \frac{1}{|S|} \sum_{s \in S} |\phi(s)|^2 = \|\phi\|^2, \end{aligned}$$

for all $\phi \in \mathbb{C}^S$. Hence we have

$$\|K(x)\| = \sup_{\substack{\|\phi\|=1 \\ \phi \in \mathbb{C}^S}} \|K(x)\phi\| = 1. \quad (3.18)$$

Let x be in E/L^* and 1 an eigenvalue of $P(x)$. Then for a corresponding eigenvector ψ , we have

$$\|\psi\|^2 = \langle \psi, \psi \rangle,$$

$$\begin{aligned}
&= \langle P(x)\psi, \psi \rangle, \\
&= \frac{1}{n+1} \langle K(x)\psi + \sum_{i=1}^n S_i\psi, \psi \rangle, \\
&= \frac{1}{n+1} \{ \langle K(x)\psi, \psi \rangle + \sum_{i=1}^n \langle S_i\psi, \psi \rangle \}. \tag{3.19}
\end{aligned}$$

But by Schwarz inequality, together with equations (3.17) and (3.18) we have

$$|\langle K(x)\psi, \psi \rangle| \leq \|K(x)\psi\| \|\psi\| \leq \|K(x)\| \|\psi\|^2 = \|\psi\|^2,$$

and for all $i = 1, \dots, n$,

$$|\langle S_i\psi, \psi \rangle| \leq \|S_i\psi\| \|\psi\| = \|S_i\| \|\psi\|^2 = \|\psi\|^2.$$

If $|\langle K(x)\psi, \psi \rangle| < \|\psi\|^2$ or $|\langle S_i\psi, \psi \rangle| < \|\psi\|^2$ for some i , $i = 1, \dots, n$, then by equation (3.19) we obtain

$$\|\psi\|^2 = \frac{1}{n+1} \left| \langle K(x)\psi, \psi \rangle + \sum_{i=1}^n \langle S_i\psi, \psi \rangle \right| < \|\psi\|^2,$$

which is a contradiction. Thus we have $|\langle K(x)\psi, \psi \rangle| = |\langle S_i\psi, \psi \rangle| = \|\psi\|^2$, for all $i = 1, \dots, n$, from which by Schwarz inequality we find

$$S_i\psi = \alpha_i\psi,$$

for all $i = 1, \dots, n$, and

$$K(x)\psi = \alpha_0\psi,$$

for some constants $|\alpha_i|$, $i = 0, \dots, n$. By equations (3.17) and (3.18), we have $|\alpha_i| \|\psi\| = \|\psi\|$, for all $i = 0, \dots, n$. Since ψ is nonzero function, we obtain $|\alpha_i| = 1$, for all $i = 0, \dots, n$. Hence from equation (3.19) we have

$$\frac{1}{n+1} \{ \alpha_0 + \dots + \alpha_n \} = 1,$$

which implies $\lambda_i = 1$, for all $i = 1, \dots, n$ (see Appendix A for a detailed proof).

We have therefore,

$$S_i\psi = \psi, \tag{3.20}$$

for all $i = 1, \dots, n$, and

$$K(x)\psi = \psi. \quad (3.21)$$

From equation (3.20), we have $\psi(ss_i) = \psi(s)$ for all s in S and $i = 1, \dots, n$, from which it immediately follows that ψ is a nonzero constant function. Also, from equation (3.21) we have

$$\psi(s) = (K(x)\psi)(s) = e^{2\pi i \langle x, sr_0 \rangle} \psi(ss_0).$$

But since ψ is non zero constant function, we have $e^{2\pi i \langle x, sr_0 \rangle} = 1$. In other words, $\langle x, sr_0 \rangle \in \mathbf{Z}$ for all $s \in S$, which implies $\langle x, r \rangle \in \mathbf{Z}$, for all $r \in \Delta$. Thus x must be an element of L^* and hence $x = 0 \pmod{L^*}$. Therefore, we conclude that as an element of E/L^* , x must be equal to 0.

□

An alternative proof for part (1) of Lemma 3.3.2 above can be derived from equations (3.17) and (3.18) by observing that

$$\begin{aligned} \|P(x)\| &= \frac{1}{n+1} \left\| K(x) + \sum_{i=1}^n S_i \right\|, \\ &\leq \frac{1}{n+1} \left\{ \|K(x)\| + \sum_{i=1}^n \|S_i\| \right\}, \\ &\leq 1. \end{aligned}$$

This implies that all eigenvalues of $P(x)$ are less or equal to one in absolute value, since $P(x)$ is a bounded self-adjoint linear operator in a complex Hilbert space.

It is worth mentioning that for every $x \in E/L^*$, $P(x)$ has finite number of eigenvalues. For every $x \in E/L^*$, let $I_x = \{0, 1, \dots, k_x\}$ be the index set of eigenvalue of $P(x)$.

Lemma 3.3.3 *There is only one eigenvalue of $P(x)$ which tends to 1 as x tends to 0.*

Proof. Consider polynomial of t given by

$$\det(tI - P(x)) = \prod_{i \in I_x} (t - \lambda_i(x)). \quad (3.22)$$

For $x = 0$ equation (3.22) becomes

$$\det(tI - P(0)) = \prod_{i \in I_0} (t - \lambda_i(0)).$$

Suppose there are more than one eigenvalues of $P(x)$ which tend to 1 as x tends to 0. Then

$$\det(tI - P(0)) = (t - 1)^r \prod_{\substack{i \in I_0 \\ i \neq 0}} (t - \lambda_i(0)), \quad r > 1.$$

But, since 1 is a simple eigenvalue of $P(0)$, it follows that $r = 1$. This is a contradiction, in other words the supposition is false.

□

For x close to 0, let $\lambda_0(x)$ be the eigenvalue of $P(x)$. The projection operator $E_0(x)$ of \mathbf{C}^S onto the eigenspace of $P(x)$ corresponding to the eigenvalue $\lambda_0(x)$ is given by

$$E_0(x) = \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} dz, \quad (3.23)$$

where C is the closed curve in \mathbf{C} with the condition that $\lambda_0(x)$ is the only eigenvalue of $P(x)$ in the interior. Indeed, if $\psi_i(x)$ is an eigenvector of $P(x)$ with an eigenvalue $\lambda_i(x)$, then

$$(zI - P(x))^{-1} \psi_i(x) = (z - \lambda_i(x))^{-1} \psi_i(x).$$

We observe now

$$\begin{aligned} E_0(x) \psi_i(x) &= \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} dz \psi_i(x), \\ &= \frac{1}{2\pi i} \oint_C (zI - P(x))^{-1} \psi_i(x) dz, \\ &= \frac{1}{2\pi i} \oint_C (z - \lambda_i(x))^{-1} dz \psi_i(x), \\ &= \begin{cases} \psi_0(x), & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $E_0(x)$ is the projection operator onto the eigenspace corresponding to the eigenvalue $\lambda_0(x)$.

The constant function 1_S defined by $1_S(s) = s$ for all s in S is an eigenvector of the operator $P(0)$ corresponding to the eigenvalue 1. Now let

$$\phi_0(x) = E_0(x)1_S.$$

Then $\phi_0(x)$ is an eigenvector of $P(x)$ with the eigenvalue $\lambda_0(x)$, for small x . Let us write $\phi_0(x, s)$ for $\phi_0(x)(s)$.

Lemma 3.3.4 *Let $\lambda_0(x)$ be the eigenvalue of $P(x)$ which tends to one as x tends to zero. Then for small x in E/L^* we have*

$$\lambda_0(x) = \lambda_0(-x).$$

Proof. Let $\phi_0(x, s)$ be the eigenvector corresponding to the eigenvalue $\lambda_0(x)$ for small x in E/L^* . We observe, from the definition of $P(x)$ in equation (3.7),

$$\begin{aligned} \overline{(P(x)\psi)(s)} &= \frac{1}{n+1} \left\{ e^{2\pi i \langle -x, s r_0 \rangle} \overline{\psi(ss_0)} + \sum_{i=1}^n \overline{\psi(ss_i)} \right\} \\ &= \overline{(P(-x)\bar{\psi})(s)}, \end{aligned}$$

for all $\psi \in \mathbf{C}^S$ and for all $s \in S$. In other words, we have $\overline{P(x)\psi} = P(-x)\bar{\psi}$, for all $\psi \in \mathbf{C}^S$. Hence for small x in E/L^* we obtain

$$\begin{aligned} \overline{\phi_0(x, s)} &= \overline{(E_0(x)1)(s)}, \\ &= (E_0(-x)1)(s), \\ &= \phi_0(-x, s). \end{aligned} \tag{3.24}$$

By equation (3.24), we observe

$$\lambda_0(x)\phi_0(x, s) = P(x)\phi_0(x, s),$$

$$\begin{aligned}
&= P(x)\overline{\phi_0(-x, s)}, \\
&= \overline{P(-x)\phi_0(-x, s)}, \\
&= \overline{\lambda_0(-x)\phi_0(-x, s)}, \\
&= \lambda_0(-x)\phi_0(x, s),
\end{aligned}$$

since $\lambda_0(-x)$ is real. We can conclude, therefore, $\lambda_0(x) = \lambda_0(-x)$, for small x in E/L^* .

□

Lemma 3.3.5 *Let $\lambda_0(x)$ be the eigenvalue of $P(x)$ which tends to one as x tends to zero. Then we have*

$$\lambda_0(x) = 1 - c\langle x, x \rangle + O(\|x\|^4),$$

where $c = \frac{4\pi^2}{n(n+1)^2}$ and $O(\|x\|^4)$ denotes terms containing fourth and higher powers of $\|x\|$.

Proof. Since for small x in E/L^* , $\lambda_0(x) = \lambda_0(-x)$ and $\lambda_0(0) = 1$, using Maclaurin's formula we find

$$\lambda_0(x) = 1 + \lambda_{0,2}(x) + O(\|x\|^4), \tag{3.25}$$

where $\lambda_{0,2}(x)$ denotes the second degree term in x and $O(\|x\|^4)$ denotes terms containing fourth or higher powers in $\|x\|$. If we write $\phi_0(x, s)$ in the form

$$\phi_0(x, s) = \alpha(x, s) + i\beta(x, s),$$

where $\alpha(x, s)$ and $\beta(x, s)$ are real and imaginary parts of $\phi_0(x, s)$ respectively, by equation (3.24) then we obtain

$$\alpha(-x, s) = \alpha(x, s),$$

and

$$\beta(-x, s) = -\beta(x, s).$$

Hence, using Maclaurin's formula, $\alpha(x, s)$ and $\beta(x, s)$ can be written in the form

$$\alpha(x, s) = 1 + \alpha_2(x, s) + O(\|x\|^4), \quad (3.26)$$

and

$$\beta(x, s) = \beta_1(x, s) + O(\|x\|^3) \quad (3.27)$$

respectively, where $\alpha_2(x, s)$ and $\beta_1(x, s)$ are the corresponding second degree and first degree terms in x of Maclaurin series of $\alpha(x, s)$ and $\beta(x, s)$ respectively, and $O(\|x\|^3)$ denotes terms containing third and higher powers of $\|x\|$.

We observe now

$$\begin{aligned} \lambda_0(x)\phi_0(x, s) &= P(x)\phi_0(x, s), \\ &= \frac{1}{n+1} \left\{ e^{2\pi i \langle x, sr_0 \rangle} \phi_0(x, ss_0) + \sum_{i=1}^n \phi_0(x, ss_i) \right\}. \end{aligned} \quad (3.28)$$

Comparing real and imaginary parts in equation (3.28), we obtain

$$\begin{aligned} \lambda_0(x)\alpha(x, s) &= \frac{1}{n+1} \left\{ \alpha(x, ss_0) \cos 2\pi \langle x, sr_0 \rangle - \right. \\ &\quad \left. \beta(x, ss_0) \sin 2\pi \langle x, sr_0 \rangle + \sum_{i=1}^n \alpha(x, ss_i) \right\}, \end{aligned} \quad (3.29)$$

and

$$\begin{aligned} \lambda_0(x)\beta(x, s) &= \frac{1}{n+1} \left\{ \beta(x, ss_0) \cos 2\pi \langle x, sr_0 \rangle + \right. \\ &\quad \left. \alpha(x, ss_0) \sin 2\pi \langle x, sr_0 \rangle + \sum_{i=1}^n \beta(x, ss_i) \right\}. \end{aligned} \quad (3.30)$$

On the other hand, multiplying equation (3.25) with equations (3.26) and (3.27) respectively, we obtain

$$\lambda_0(x)\alpha(x, s) = 1 + \lambda_{0,2}(x) + \alpha_2(x, s) + O(\|x\|^4), \quad (3.31)$$

and

$$\lambda_0(x)\beta(x, s) = \beta_1(x, s) + O(\|x\|^3). \quad (3.32)$$

Using equation (3.31) and comparing second degree of equation (3.29), we find

$$\lambda_{0,2}(x) + \alpha_2(x, s) = \frac{1}{n+1} \left\{ \alpha_2(x, ss_0) - \frac{1}{2}(2\pi\langle x, sr_0 \rangle)^2 - 2\pi\langle x, sr_0 \rangle\beta_1(x, ss_0) + \sum_{i=1}^n \alpha_2(x, ss_i) \right\}. \quad (3.33)$$

Using equation (3.32) and comparing first degree of equation (3.30), we find

$$\beta_1(x, s) = \frac{1}{n+1} \left\{ \beta_1(x, ss_0) + 2\pi\langle x, sr_0 \rangle + \sum_{i=1}^n \beta_1(x, ss_i) \right\}. \quad (3.34)$$

We shall now evaluate $\lambda_0(x)$. Using Schur's orthogonality relations (see [26]), we obtain

$$\frac{1}{|S|} \sum_{s \in S} \langle x, sr_0 \rangle^2 = \frac{1}{n} \langle r_0, r_0 \rangle \langle x, x \rangle.$$

We recall from Riesz's representation theorem on Hilbert spaces that there exists a unique vector $b_s \in E$ such that

$$\beta_1(x, s) = \langle b_s, x \rangle. \quad (3.35)$$

Let us make the assumption that $\langle b_s, x \rangle = \langle sb, x \rangle$, for a fixed vector $b \in E$, which will be justified later. Then by equation (3.35),

$$\beta_1(x, ss_0) = \langle ss_0 b, x \rangle.$$

By Schur's orthogonality relations, we observe

$$\begin{aligned} \frac{1}{|S|} \sum_{s \in S} \langle x, sr_0 \rangle \langle ss_0 b, x \rangle &= \frac{1}{|S|} \sum_{s \in S} \langle s^{-1}x, r_0 \rangle \langle s^{-1}x, s_0 b \rangle, \\ &= \frac{1}{n} \langle r_0, s_0 b \rangle \langle x, x \rangle. \end{aligned} \quad (3.36)$$

Thus if equation (3.33) is summed up for all $s \in S$, we obtain

$$\begin{aligned} \lambda_{0,2}(x) &= \frac{1}{n+1} \left\{ -\frac{2\pi^2}{n} \langle r_0, r_0 \rangle \langle x, x \rangle - \frac{2\pi}{n} \langle r_0, s_0 b \rangle \langle x, x \rangle \right\}, \\ &= -c \langle x, x \rangle, \end{aligned} \quad (3.37)$$

where c is a constant given by

$$c = \frac{2\pi}{n(n+1)} (2\pi + \langle r_0, s_0 b \rangle). \quad (3.38)$$

We shall now evaluate $\langle s_0 b, r_0 \rangle$. From equation (3.34) and (3.35), given $x \in E/L^*$,

$$\begin{aligned}\langle sb, x \rangle &= \frac{1}{n+1} \left\{ 2\pi \langle x, sr_0 \rangle + \sum_{i=0}^n \langle ss_i b, x \rangle \right\}, \\ &= \frac{1}{n+1} \left\{ \langle s \left(2\pi r_0 + \sum_{i=0}^n s_i b \right), x \rangle \right\}.\end{aligned}$$

Hence we find

$$sb = \frac{1}{n+1} \left\{ s \left(2\pi r_0 + \sum_{i=0}^n s_i b \right) \right\}, \quad (3.39)$$

from which it follows immediately

$$b = \frac{1}{n+1} \left\{ 2\pi r_0 + \sum_{i=0}^n s_i b \right\}. \quad (3.40)$$

By equation (2.3) in Chapter 2 Section 2.2, we have

$$s_i b = b - \langle r_i, b \rangle r_i, \quad (3.41)$$

for all $i = 0, \dots, n$. Using equation (3.40) and substituting $s_i b$ by equation (3.41) we observe

$$\begin{aligned}b &= \frac{1}{n+1} \left\{ 2\pi r_0 + \sum_{i=0}^n (b - \langle r_i, b \rangle r_i) \right\}, \\ &= \frac{1}{n+1} \left\{ 2\pi r_0 + (n+1)b - \sum_{i=0}^n \langle r_i, b \rangle r_i \right\}.\end{aligned}$$

We obtain thus

$$\sum_{i=0}^n \langle r_i, b \rangle r_i = 2\pi r_0,$$

and hence collecting variable r_0 , we get

$$\sum_{i=1}^n \langle r_i, b \rangle r_i = (2\pi - \langle r_0, b \rangle) r_0.$$

Since $r_0 = \sum_{i=1}^n r_i$ and $\{r_1, r_2, \dots, r_n\}$ is an independent linear set, we find

$$\langle r_i, b \rangle = 2\pi - \langle r_0, b \rangle, \quad (3.42)$$

for all $i = 1, \dots, n$. Furthermore, if we sum up equation (3.42) for all $i = 1, \dots, n$, we obtain

$$\begin{aligned}\langle r_0, b \rangle &= \left\langle \sum_{i=1}^n r_i, b \right\rangle, \\ &= (2\pi - \langle r_0, b \rangle)n.\end{aligned}$$

Hence solving for $\langle r_0, b \rangle$, we have

$$\langle r_0, b \rangle = \frac{2\pi n}{n+1}. \quad (3.43)$$

Since s_0 is an orthogonal transformation and $s_0 r_0 = -r_0$, from equation (3.43) we obtain

$$\langle r_0, s_0 b \rangle = -\langle s_0 r_0, s_0 b \rangle = -\frac{2\pi n}{n+1}.$$

Hence the constant c given in equation (3.38) becomes

$$c = \frac{2\pi}{n(n+1)} \left(2\pi - \frac{2\pi n}{n+1} \right) = \frac{4\pi^2}{n(n+1)^2}. \quad (3.44)$$

Therefore by equations (3.25) and (3.37) together with equation (3.44) we obtain

$$\lambda_0(x) = 1 - \frac{4\pi^2}{n(n+1)^2} \langle x, x \rangle + O(\|x\|^4).$$

We shall now show that our assumption that $\beta_1(x, s) = \langle sb, x \rangle$ is valid. For a fixed x let us write

$$\beta_1(x, s) = h(s).$$

Then by equation (3.34) we have

$$h(s) = \frac{1}{n+1} \left\{ k(s) + \sum_{i=0}^n h(ss_i) \right\},$$

where $k(s) = 2\pi \langle x, sr_0 \rangle$, from which we obtain

$$\{(I - P(0))h\}(s) = \frac{1}{n+1} k(s). \quad (3.45)$$

In the closed subspace $\{1_S\}^\perp$ of the annihilator of $\{1_S\}$, equation (3.45) obviously has a unique solution for h , since the operator $(I - P(0))$ is invertible in $\{1_S\}^\perp$.

To show the validity of our assumption, it is enough to show that $\langle sb, x \rangle$ and h lie in the subspace $\{1_S\}^\perp$, since s is a linear transformation on E , for each s in S . For this purpose, for a fixed x , let us consider

$$f(s) = \langle sb, x \rangle.$$

To show that $f(s) \in (1_S)^\perp$, we shall show that $\sum_{s \in S} f(s) = 0$. Let $y \in E$ and $i \in \{1, \dots, n\}$. By equation (2.3), we have

$$y = s_i y + \langle r_i, y \rangle r_i.$$

Hence we obtain

$$sy = ss_i y + \langle r_i, y \rangle sr_i,$$

for all s in S , from which it follows that

$$\sum_{s \in S} \langle r_i, y \rangle sr_i = 0.$$

Since y and i is chosen arbitrarily, we find

$$\sum_{s \in S} sr_i = 0,$$

for all $i = 0, \dots, n$. Since $\{r_0, \dots, r_n\}$ generates E , there exist real numbers η_0, \dots, η_n such that

$$b = \eta_0 r_0 + \eta_1 r_1 + \eta_2 r_2 + \dots + \eta_n r_n.$$

Therefore, we obtain

$$\sum_{s \in S} f(s) = \sum_{s \in S} \langle sb, x \rangle = \sum_{i=0}^n \eta_i \sum_{s \in S} \langle sr_i, x \rangle = 0.$$

We shall now show that h defined by $h(s) = \beta_1(x, s)$, for all s in S and for a fixed x , is in the space $(1_S)^\perp$. Let $\{E_i(x) : x \in I_x\}$ be the set of projection operators to the eigenspaces corresponding to the operator $P(x)$. Then we have

$$I = E_0(x) + \sum_{i \neq 0} E_i(x),$$

where I is the identity operator and $E_0(x)$ is the projection operator corresponding to the eigenvalue $\lambda_0(x)$. Hence

$$1_S = I1_S = E_0(x)1_S + \sum_{i \neq 0} E_i(x)1_S. \quad (3.46)$$

The inner product of each side of equation (3.46) with $\phi_0(x, s) = E_0(x)1_S$ is given by

$$\langle 1_S, \phi_0(x, s) \rangle = \langle \phi_0, \phi_0(x, s) \rangle,$$

since $\langle \phi_0, \phi_i(x, s) \rangle = 0$, for all $i \neq 0$. Hence $\langle 1_S, \phi_0(x, s) \rangle$ is a real number. On the other hand,

$$\begin{aligned} \langle 1_S, \phi_0(x, s) \rangle &= \frac{1}{|S|} \sum_{s \in S} \phi_0(x, s), \\ &= \frac{1}{|S|} \sum_{s \in S} (\alpha(x, s) + i\beta(x, s)), \end{aligned}$$

Hence we have $\sum_{s \in S} \beta(x, s) = 0$ from which comparing first degree terms in x we get,

$$\sum_{s \in S} \beta_1(x, s) = 0.$$

In other words the function h is in the space $(1_S)^\perp$. Hence $\beta_1(x, s)$ and $\langle sb, x \rangle$ both satisfy equation (3.45). But the solution of (3.45) should be unique in the space $(1_S)^\perp$. Therefore we conclude that $\beta_1(x, s)$ can be written as $\langle sb, x \rangle$ for a vector b in E .

□

3.4 Evaluation of the Ultimate Behaviour of the Walk

In Chapter 3 we obtained that the probability of the particle arriving at the origin after m steps is given by an integral expression. This integral is just the inverse

Fourier transform of the convolution function $\varepsilon_{e_A} \times \mu^{\times m}$ at (x, e_S) and can be expressed as an integral of the inner product involving the operator $P(x)$ given in Lemma 3.4.1. Our main theorem in this section is given in Theorem 3.4.2 which is straight forward from Lemma 3.4.1.

Lemma 3.4.1 *The probability of the particle arriving at the origin after m steps is given by*

$$\mu^{\times m}(0, e_S) = |S| \int_{E/L^*} \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \quad (3.47)$$

Proof. Using equation (3.6), for all x in E/L^* and s in S , we have

$$P(x) \varepsilon_{e_A}^\wedge(x, s) = (\varepsilon_{e_A} \times \mu)^\wedge(x, s).$$

For any positive integer m ,

$$P(x)^m \varepsilon_{e_A}^\wedge(x, s) = (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, s).$$

In particular, for $s = e_S$, we find

$$P(x)^m \varepsilon_{e_A}^\wedge(x, e_S) = (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, e_S).$$

But the Fourier transform of $\varepsilon_{e_A}(v, s)$ is given by

$$\varepsilon_{e_A}^\wedge(x, s) = \sum_{v \in L} \varepsilon_{e_A}(v, s) e^{-2\pi i \langle x, v \rangle},$$

for all x in E/L^* and s in S , from which it follows that for all x in E/L^* ,

$$\varepsilon_{e_A}^\wedge(x, s) = \begin{cases} 1, & \text{if } s = e_S, \\ 0, & \text{otherwise.} \end{cases} \quad (3.48)$$

In other words, as a function on S , $\varepsilon_{e_A}^\wedge(x) = \varepsilon_{e_S}$, where ε_{e_S} is an indicator function at the identity e_S defined on S . Therefore

$$\begin{aligned} (P(x)^m \varepsilon_{e_S})(e_S) &= P(x)^m \varepsilon_{e_A}^\wedge(x, e), \\ &= (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, e_S). \end{aligned}$$

Hence from equation (3.4) we obtain

$$\mu^{\times m}(0, e_S) = \int_{E/L^*} (\varepsilon_{e_A} \times \mu^{\times m})^\wedge(x, e_S) dx = \int_{E/L^*} (P(x)^m \varepsilon_{e_S})(e_S) dx. \quad (3.49)$$

Now by the definition of the inner product \langle, \rangle on \mathbf{C}^S , we have

$$\begin{aligned} \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle &= \frac{1}{|S|} \sum_{s \in S} (P(x)^m \varepsilon_{e_S})(s) \overline{\varepsilon_{e_S}(s)}, \\ &= \frac{1}{|S|} (P(x)^m \varepsilon_{e_S})(e_S). \end{aligned}$$

Hence equation (3.49) can be written in the form

$$\mu^{\times m}(0, e_S) = |S| \int_{E/L^*} \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx.$$

□

In order to estimate $\mu^{\times m}(0, e_S)$ we recall from basic fact in linear operator theory that for every bounded linear operator T on a Banach space X with $\|T\| < 1$, it follows that $(I - T)^{-1}$ exists as a bounded linear operator on X and

$$(I - T)^{-1} = \sum_{m=0}^{\infty} T^m.$$

Using this fact together with Lemma 3.4.1 we have the following theorem.

Theorem 3.4.2 *The expected number of times the particle visits the origin is given by*

$$\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) = \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx.$$

Proof. Let x be an element of E/L^* , and $|\theta| < 1$. Then $(I - \theta P(x))$ is invertible and its inverse is given by

$$(I - \theta P(x))^{-1} = I + \theta P(x) + \theta^2 P(x)^2 + \dots \quad (3.50)$$

Using monotone convergence theorem, the expected number of times the particle visits the origin is given by the series

$$\begin{aligned}
\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) &= \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \theta^m \mu^{\times m}(0, e_S), \\
&= \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} |S| \int_{E/L^*} \theta^m \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\
&= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \sum_{m=0}^{\infty} \langle \theta^m P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\
&= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle \sum_{m=0}^{\infty} \theta^m P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\
&= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \quad (3.51)
\end{aligned}$$

Alternatively, we can prove Theorem 3.4.2 as follows. From the previous calculation we have

$$\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) = \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} |S| \int_{E/L^*} \theta^m \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx.$$

Since for all x in E/L^* , by Schwarz inequality,

$$|\langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle| \leq \|P(x)\|^m \|\varepsilon_{e_S}\| \leq \frac{1}{|S|} < 1,$$

it follows that

$$\begin{aligned}
\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) &= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \sum_{m=0}^{\infty} \theta^m \langle P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\
&= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle \sum_{m=0}^{\infty} \theta^m P(x)^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \quad (3.52)
\end{aligned}$$

Now let, for p in \mathbf{N} , $B_p = \sum_{m=0}^p \theta^m P(x)^m$. Then for $k > l$ we have

$$B_k - B_l = \sum_{m=l+1}^k \theta^m P(x)^m$$

Hence we obtain

$$\|B_k - B_l\| \leq \sum_{m=l+1}^k \theta^m \|P(x)\|^m.$$

Since $\|P(x)\| \leq 1$, it follows that $\{B_k\}$ is a Cauchy sequence and hence it is convergent. Let

$$\lim_{k \rightarrow \infty} B_k = B.$$

Then we obtain

$$\begin{aligned} B(I - \theta P(x)) &= \sum_{m=0}^{\infty} \theta^m P(x)^m (I - \theta P(x)), \\ &= \sum_{m=0}^{\infty} \theta^m P(x)^m - \sum_{m=0}^{\infty} \theta^{m+1} P(x)^{m+1}, \\ &= I. \end{aligned}$$

Hence we have

$$(I - \theta P(x))^{-1} = \sum_{m=0}^{\infty} \theta^m P(x)^m,$$

for all x in E/L^* . Therefore, equation (3.52) becomes

$$\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) = \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx,$$

which completes the proof. □

The convergence of the integral in the right hand side of equation (3.51) depends on the behavior of $(I - P(x))^{-1}$. The integral can be expressed as the sum of two integrals, the first an integral over E/L^* with $\|x\| \geq \delta$ for some small positive real numbers, and the second an integral over E/L^* with $\|x\| < \delta$. Since for $x \neq 0$, $(I - P(x))$ is invertible and $(I - P(x))^{-1}$ is bounded, it follows that the first integral is convergent. We will now investigate the behavior of the second integral.

Lemma 3.4.3 *Let δ be a small positive real number. Then*

$$\begin{aligned} &\lim_{\theta \uparrow 1} \int_{\|x\| < \delta} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx \\ &= \sum_{i \neq 0} \int_{\|x\| < \delta} \langle (1 - \lambda_i(x))^{-1} E_i(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx + \int_{\|x\| < \delta} \frac{|\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2}{\langle x, x \rangle (c + O(\|x\|^2))} dx, \end{aligned}$$

where c is a constant given by equation (3.44), and $O(\|x\|^2)$ denotes terms containing second or higher powers of $\|x\|$, and $\phi(x, s)$ is the eigenvector corresponding to the eigenvalue $\lambda_0(x)$.

Proof. Let $|\theta| < 1$ and $\{\lambda_i(x) : i \in I_x\}$ the set of eigenvalues of $P(x)$ with $\{E_i(x) : i \in I_x\}$ the corresponding orthogonal projections from the Hilbert space \mathbb{C}^S to the corresponding eigenspaces. From equation (3.50) we observe that

$$\begin{aligned} (I - \theta P(x))^{-1} &= \sum_{m=1}^{\infty} \theta^m P(x)^m, \\ &= \sum_{m=1}^{\infty} \sum_{i \in I_x} \theta^m \lambda_i(x)^m E_i(x), \\ &= \sum_{i \in I_x} \sum_{m=1}^{\infty} \theta^m \lambda_i(x)^m E_i(x), \\ &= \sum_{i \in I_x} (1 - \theta \lambda_i(x))^{-1} E_i(x). \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\substack{\|x\| < \delta \\ E/L^*}} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx \\ &= \int_{\substack{\|x\| < \delta \\ E/L^*}} \langle (1 - \theta \lambda_0(x))^{-1} E_0(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx \\ &\quad + \int_{\substack{\|x\| < \delta \\ E/L^*}} \langle \sum_{i \neq 0} (1 - \theta \lambda_i(x))^{-1} E_i(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\ &= \int_{\substack{\|x\| < \delta \\ E/L^*}} \langle (1 - \theta \lambda_0(x))^{-1} E_0(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx \\ &\quad + \sum_{i \neq 0} \int_{\substack{\|x\| < \delta \\ E/L^*}} \langle (1 - \theta \lambda_i(x))^{-1} E_i(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \end{aligned} \tag{3.53}$$

To complete the proof we shall now examine the integral in the first term in the right hand side of equation (3.53). Let $\phi_0(x, s)$ be an eigenvector corresponding to eigenvalue $\lambda_0(x)$. Then

$$E_0(x) \varepsilon_{e_S} = \langle \varepsilon_{e_S}, \phi_0(x, s) \rangle \phi_0(x, s). \tag{3.54}$$

Using equation (3.54), we observe

$$\int_{\substack{\|x\| < \delta \\ E/L^*}} \langle (1 - \theta \lambda_0(x))^{-1} E_0(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx$$

$$\begin{aligned}
&= \int_{E/L^\bullet}^{\|x\| < \delta} (1 - \theta \lambda_0(x))^{-1} \langle \varepsilon_{e_S}, \phi_0(x, s) \rangle \langle \phi_0(x, s), \varepsilon_{e_S} \rangle dx, \\
&= \int_{E/L^\bullet}^{\|x\| < \delta} (1 - \theta \lambda_0(x))^{-1} |\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2 dx, \\
&= \int_{E/L^\bullet}^{\|x\| < \delta} \frac{1}{1 - \theta(1 - c\langle x, x \rangle + O(\|x\|^4))} |\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2 dx,
\end{aligned}$$

where c is given by (3.44). Hence we have

$$\begin{aligned}
&\lim_{\theta \uparrow 1} \int_{E/L^\bullet}^{\|x\| < \delta} \langle (I - \theta P(x))^{-1} \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx \\
&= \sum_{i \neq 0} \int_{E/L^\bullet}^{\|x\| < \delta} \langle (1 - \lambda_i(x))^{-1} E_i(x) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx + \int_{E/L^\bullet}^{\|x\| < \delta} \frac{|\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2}{\langle x, x \rangle (c + O(\|x\|^2))} dx. \quad (3.55)
\end{aligned}$$

Hence by equations (3.53) and (3.55), Lemma 3.4.3 follows. \square

Theorem 3.4.4 *The expected number of times the particle visits the origin is finite if $n > 2$ and infinite if $n = 2$.*

Proof. Again, all terms inside the sum in the right hand side of equation (3.55) are bounded, since $\lambda_0(x)$ is the only eigenvalue which tends to one as x tends to zero. Hence the convergence of the integral in the left hand side of equation (3.55) depends only on the integral

$$\int_{E/L^\bullet}^{\|x\| < \delta} \frac{|\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2}{\langle x, x \rangle (c + O(\|x\|^2))}.$$

Moreover, as x tends to zero, $|\langle \varepsilon_{e_S}, \phi_0(x, s) \rangle|^2$ tends to $|\langle \varepsilon_{e_S}, 1 \rangle|^2 = \frac{1}{|S|^2}$. Hence the convergence of this integral depends on the denominator of the integrand. But $c + O(\|x\|^2)$ is bounded and tends to c as x tends to zero. As a result, $\sum_{m=0}^{\infty} \mu(0, e_S)$ is infinite or finite according to the value of the integral

$$\int_{E/L^\bullet}^{\|x\| < \delta} \frac{1}{\langle x, x \rangle} dx,$$

which is infinite for $n = 2$ and finite for $n > 2$. We conclude therefore

Corollary 3.4.5 *The ultimate behaviour of the walk is recurrent if $n = 2$ and transitory if $n > 2$.*

Proof. Case 1: $n > 2$.

Here we have that the expected number of times the particle visits the identity is finite. This means that the actual number of visits to the identity is finite with probability one. This result is true for any element a in A , hence we have that for any $R < \infty$, the particle ultimately stops visiting the chambers within a distance R of e_A , which says that the random walk is transitory.

Case 2: $n = 2$.

Suppose that the probability of the particle ultimately return to the identity is p . Then the probability of visiting the identity for at least m times is p^{m-1} , including the visit at time $m = 0$. As a consequence, the probability of visiting the identity for exactly m times is given by

$$p^{m-1} - p^m = p^{m-1}(1 - p).$$

As a result, if $p < 1$, the expected number of visits is

$$\begin{aligned} &= \sum_{m=1}^{\infty} mp^{m-1}(1 - p), \\ &= (1 - p)^{-1} < \infty. \end{aligned}$$

But this is a contradiction, since we have shown in Theorem 3.4.4 that the expected number of visits is infinite. Hence the particle visits the identity infinitely many times with probability one. Therefore, we conclude that the random walk is recurrent for the case $n = 2$.

□

Chapter 4

Generalization

In Chapter 3 we have discussed a particular random walk and obtained the criterion for the ultimate behaviour of the walk. The content of this chapter is a generalization of our result for a random walk with a general probability measure μ under the assumption that the support of μ generates the group A . Since the technique in this generalization come from the theory of group representation, we devote Section 4.1 to discuss some general results of the representation theory of groups. In Subsection 4.1.1 we use the method of “little groups” introduced by Wigner and Mackey to construct all irreducible unitary representations of the group A . Finally, the generalization of our result is given in section 4.2

4.1 Representations of Groups

Let G be a separable locally compact topological group and \mathcal{H} be a Hilbert space over the scalar field \mathbf{C} of complex numbers. A unitary representation of G in \mathcal{H} is a homomorphism ρ of G into the space of all unitary operators from the Hilbert space \mathcal{H} into itself. In other words,

$$\rho(g_1 g_2) = \rho(g_1) \rho(g_2), \quad g_1, g_2 \in G.$$

A unitary representation ρ is said to be continuous if the mapping

$$g \longrightarrow \rho(g)v, \quad g \in G$$

from G to \mathcal{H} is continuous for all v in \mathcal{H} . The space \mathcal{H} is called the representation space and the operators $\rho(g)$ representation operators. The dimension $d(\mathcal{H})$ of \mathcal{H} is called the degree of the representation ρ .

In this chapter, for brevity, we shall simply say locally compact group for separable locally compact topological group, and unitary representation for continuous unitary representation.

A unitary representation ρ of G in \mathcal{H} is called irreducible if $\mathcal{H} \neq \{0\}$ and there is no proper closed subspace of \mathcal{H} invariant under the action of representation operators $\rho(g)$, for all g in G .

Unitary representations ρ_1 and ρ_2 of G in Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively are said to be equivalent if there exists a unitary isomorphism τ from \mathcal{H}_1 onto \mathcal{H}_2 such that

$$\tau \rho_1(s) = \rho_2(s) \tau,$$

for all $s \in G$.

Let G be a finite group. Left regular representations L and right regular representation R of G in \mathbb{C}^G defined by

$$(L(s)f)(u) = f(s^{-1}u), \tag{4.1}$$

$$(R(s)f)(u) = f(us), \tag{4.2}$$

respectively, for all f in \mathbb{C}^G and s, u in G .

Let n_ρ be the degree of the irreducible representation ρ of G . Then a map from $\mathbb{C}[G]$ into $\prod M_{n_\rho}(\mathbb{C})$ given in matrix form by

$$\tilde{f}(\rho) = \rho(f) \tag{4.3}$$

is the Fourier transform of the function f . Any representation ρ of G in \mathbf{C}^G extends to a representation of $\mathbf{C}[G]$ by the rule

$$\rho(f) = \sum_{s \in G} f(s) \rho(s). \quad (4.4)$$

Let G be a locally compact group and K an open and closed subgroup of G so that the index $[G : K]$ of K in G is denumerable and discrete. Let σ be a unitary representation of K in a Hilbert space \mathcal{K} . Let $L^2(G/K, \sigma)$ be the Hilbert space of all functions f on G with values in \mathcal{K} such that

1. $f(kg) = \sigma(k)f(g)$, for all $k \in K$ and $g \in G$, and
2. $\|f\|^2 = \sum_{g \in G/K} \langle f(g), f(g) \rangle < \infty$.

The operator ρ_{g_0} acts on the vector space $L^2(G/K, \sigma)$ according to the rule

$$(\rho_{g_0} f)(g) = f(gg_0),$$

for all $g_0, g \in G, f \in L^2(G, \sigma)$. Then ρ defines a unitary representation of G and it is said to be a representation of G in $L^2(G/K, \sigma)$ induced by the representation σ of K in \mathcal{K} . We may write this representation with $Ind_K^G \sigma$.

4.1.1 A Construction of Irreducible Representations of the Group A

If the group G is a semidirect product of a normal abelian subgroup K and a group H , we can construct all irreducible representations of G as an induced representation of irreducible representation of certain subgroup of H , by the method of “little groups” of Wigner and Mackey and the representations of G are unique (up to isomorphism) (see [26]).

It can be easily seen that this method can be generalized to an infinite group G where the index $[G : K]$ of K in G is finite. From Lemma 2.3.1 (1) and 2.3.2

(1), we have

$$A = TS \simeq LS_{n+1},$$

where S_{n+1} is the symmetric group of all permutations of $n+1$ elements. Hence A is a countable group and if we endow the group A with the discrete topology, then A is a locally compact group. Since L is a normal abelian group, all irreducible unitary representations of L are of degree one and form a group \mathbf{X} given by

$$\mathbf{X} = \{\chi_x : x \in E, \chi_x(v) = e^{2\pi i \langle x, v \rangle}, \text{ for all } v \in L\}.$$

The group A acts on \mathbf{X} by

$$(a\chi_x)(v) = \chi_x(ava^{-1}), \quad (4.5)$$

for all a in A , v in L , χ_x in \mathbf{X} . We note that the multiplication ava^{-1} is a multiplication in the group A . Since L is a normal subgroup, the element ava^{-1} is in L , so that equation (4.5) has sense. In particular, for all s in S_{n+1} , by Lemma 2.3.1 (3) in Chapter 2, we have

$$svs^{-1} = ((0, s)(v, e_S))(0, s)^{-1} = (sv, s)(0, s^{-1}) = (sv, e_S) = sv.$$

Let S^x be the stabilizer of χ_x in S_{n+1} ,

$$S^x = \{s \in S_{n+1} : s\chi_x = \chi_x\}.$$

Using the method of “little groups” introduced by Wigner and Mackey, we obtain that all irreducible representation ρ_x of A are of the form

$$\rho_x = \text{Ind}_{LS^x}^A(\chi_x \otimes \sigma), \quad (4.6)$$

where \otimes denotes the tensor product of the representations, $\chi_x \in \mathbf{X}$ and σ is an irreducible representation of S^x .

If x is not in L^* , then $S^x = \{e_A\}$. In fact, if s is an element of S^x , then by the definition of S^x , we have

$$\chi_x(v) = (s\chi_x)(v) = \chi_x(svs^{-1}) = \chi_x(sv).$$

Hence by the definition of the homomorphism χ_x , we obtain for all v in L

$$e^{2\pi i \langle x, v \rangle} = e^{2\pi i \langle x, sv \rangle}.$$

This implies that for all v in L we have

$$e^{2\pi i (\langle x, v \rangle - \langle x, sv \rangle)} = 1,$$

from which we find

$$\langle sx - x, sv \rangle = \langle sx, sv \rangle - \langle x, sv \rangle = \langle x, v \rangle - \langle x, sv \rangle \in \mathbf{Z},$$

for all v in L . Hence $sx - x$ is an element of \mathbf{Z} . But since x is not in E/L^* , it follows that s must be equal to e_A . We have therefore $S^x = \{e_A\}$. Hence the only representation of S^x is the identity representation 1. Equation (4.6) therefore can be written in the form

$$\rho_x = \text{Ind}_{L\{e\}}^A(\chi_x \otimes 1) = \text{Ind}_L^A \chi_x.$$

Since the group A is isomorphic to the semidirect products of the groups L and S_{n+1} , and the group S_{n+1} is of finite order, we have

$$\|f\| = \sum_{x \in A/L} \langle f(x), f(x) \rangle = \sum_{s \in S_{n+1}} \langle f(s), f(s) \rangle < \infty,$$

for all f in $L^2(A/L, \chi_x)$. Hence the representation space of ρ_x is the Hilbert space $L^2(A/L, \chi_x)$ of all complex valued functions f on A such that

$$f(v, s) = \chi_x(v) f(0, s), \text{ for all } v \in L, s \in S_{n+1}. \quad (4.7)$$

It is worth mentioning that the Hilbert space $L^2(A/L, \chi_x)$ is of finite dimension. The dimension $\dim(L^2(A/L, \chi_x))$ of $L^2(A/L, \chi_x)$ is equal to the index $[A : L]$ of L in A . Since A is isomorphic to the semidirect products LS_{n+1} of L and S_{n+1} , we have

$$\dim(L^2(A/L, \chi_x)) = [A : L] = |S_{n+1}|.$$

In fact, we observe that for a fixed x in E/L^* ,

$$f(v, s) = \chi_x(v)f(s) = e^{2\pi i \langle x, v \rangle} f(s),$$

for all f in $L^2(A/L, \chi_x)$. Hence if we define functions ϵ_t^x on A by

$$\epsilon_t^x(v, s) = \begin{cases} e^{2\pi i \langle x, tv \rangle}, & \text{if } s = t, \\ 0, & \text{otherwise,} \end{cases}$$

for all (v, s) in A , then we find that the set $\{\epsilon_t^x : t \in S_{n+1}\}$ is an orthogonal basis for $L^2(A/L, \chi_x)$. Indeed, we have $\epsilon_t^x(v, s) = e^{2\pi i \langle x, v \rangle} \epsilon_t^x(s) = \chi_x(v) \epsilon_t^x(s)$ and

$$\langle \epsilon_{t_1}^x, \epsilon_{t_2}^x \rangle = \frac{1}{|S_{n+1}|} \sum_{s \in S_{n+1}} \epsilon_{t_1}^x(s) \overline{\epsilon_{t_2}^x(s)} = \begin{cases} \frac{1}{|S_{n+1}|}, & \text{if } t_1 = t_2, \\ 0, & \text{if } t_1 \neq t_2. \end{cases}$$

Hence for x not in L^* the Hilbert space $L^2(A/L, \chi_x)$ is isomorphic to the Hilbert space $\mathbb{C}^{S_{n+1}}$. Hence the representation operator $\rho_x(a)$ can be considered as an operator on the space $\mathbb{C}^{S_{n+1}}$.

For all a in A , the representation operator $\rho_x(a)$ acts on $\mathbb{C}^{S_{n+1}}$ as follows. Let f be in $\mathbb{C}^{S_{n+1}}$, v in L and s, t in S_{n+1} . Then

$$\begin{aligned} (\rho_x(v, e_S)f)(s) &= (\rho_x(v, e_S)f)(0, s), \\ &= f((0, s)(v, e_S)), \\ &= f(sv, s), \\ &= e^{2\pi i \langle x, sv \rangle} f(0, s), \\ &= e^{2\pi i \langle x, sv \rangle} f(s), \end{aligned}$$

and

$$(\rho_x(0, t)f)(s) = f((0, s)(0, t)) = f(0, st) = f(st). \quad (4.8)$$

Hence for all $a = (v, t)$ in A ,

$$(\rho_x(a)f)(s) = (\rho_x(v, t)f)(0, s),$$

$$\begin{aligned}
&= (\rho_x(v, e_S)(0, t)f)(0, s), \\
&= (\rho_x(v, e)f)(st), \\
&= e^{2\pi i \langle x, sv \rangle} f(st),
\end{aligned}$$

for all s in S_{n+1} .

We summarize here that for each x not in L^* we have an irreducible representation ρ_x of A which acts on the Hilbert space $\mathbf{C}^{S_{n+1}}$ according to the rule

$$(\rho_x(v, t)f)(s) = e^{2\pi i \langle x, sv \rangle} f(st), \quad (4.9)$$

for every f in $\mathbf{C}^{S_{n+1}}$, v in L and s, t in S_{n+1} .

The representation operators $\rho_x(a)$ and $\rho_{tx}(a)$, x in E/L^* , a in A and t in S_{n+1} , are related in a simple fashion given in the following lemma.

Lemma 4.1.1 *For all t in S_{n+1} and a in A we have*

$$\rho_{tx}(a) = L(t)\rho_x(a)L(t^{-1}),$$

where L is left regular representation of S_{n+1} .

Proof. Let t be in S_{n+1} , $a = (v', s')$ in A . Then for all f in $\mathbf{C}^{S_{n+1}}$ we observe

$$\begin{aligned}
(L(t)\rho_x(a)L(t^{-1})f)(s) &= (L(t)\rho_x(v', s')L(t^{-1})f)(t^{-1}s), \\
&= (\rho_x(v', s')L(t^{-1})f)(t^{-1}s), \\
&= e^{2\pi i \langle x, t^{-1}sv' \rangle} (L(t^{-1})f)(t^{-1}ss'), \\
&= e^{2\pi i \langle tx, sv' \rangle} f(ss'), \\
&= (\rho_{tx}(v', s')f)(s), \\
&= (\rho_{tx}(a)f)(s).
\end{aligned}$$

Hence we find

$$(L(t)\rho_x(a)L(t^{-1})f)(s) = (\rho_{tx}(a)f)(s).$$

Since this is true for all s in S_{n+1} and f in $C^{S_{n+1}}$, we have

$$\rho_{tx}(a) = L(t)\rho_x(a)L(t^{-1}).$$

for all a in A and t in S_{n+1} .

□

4.2 Generalization

In Chapter 3 we have discussed a random walk problem with symmetric transition probability on the collection of chambers \mathcal{C} . We will now generalize our work to a random walk on it with a general probability measure.

As defined in Chapter 3, we think of a particle moving on the collection of chambers \mathcal{C} . The particle starts at time 0 at the chamber C_0 and moves at time $t \geq 1$ to one of adjacent chambers in \mathcal{C} with general transition probability measure μ such that the support $Supp(\mu)$ of μ

$$Supp(\mu) = \{a \in A : \mu(a) \neq 0\}$$

generates A . Here we have $\sum_{a \in A} \mu(a) = \sum_{a \in Supp(\mu)} \mu(a) = 1$. The steps are statistically independent of the preceding steps.

Our generalization of Lemma 3.4.1 is given in the following lemma.

Lemma 4.2.1 *The probability of the particle visiting the origin after m steps is given by*

$$\mu^{\times m}(0, e_S) = |S_{n+1}| \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx,$$

for all s in S_{n+1} .

Proof. The probability of the particle visiting the origin after m steps is exactly $\mu^{\times m}(0, e_S)$ and by Lemma 3.2.1 we have

$$\mu^{\times m}(0, e_S) = \int_{E/L^*} (\mu^{\times m} \times \varepsilon_{e_A})^\wedge(x, e_S) dx, \quad (4.10)$$

where $(\mu^{\times m} \times \varepsilon_{e_A})^\wedge(x, s)$ is the Fourier transform of the function $(\mu^{\times m} \times \varepsilon_{e_A})(v, s)$ with respect to the variable v in L . Thus we have

$$\mu^{\times m}(0, e_S) = \int_{E/L^\star} (\mu^{\times m})^\wedge(x, e_S) \varepsilon_{e_S}^\wedge(x, e_S) dx.$$

Using equation (3.48) in page (43), we obtain

$$\begin{aligned} \mu^{\times m}(0, e_S) &= \int_{E/L^\star} (\mu^{\times m})^\wedge(x, e_S) dx, \\ &= \int_{E/L^\star} \sum_{v \in L} (\mu^{\times m})(v, e_S) e^{-2\pi i \langle x, v \rangle} dx. \end{aligned}$$

Changing variables of integration from x to $-x$ we find

$$\begin{aligned} \mu^{\times m}(0, e_S) &= \int_{E/L^\star} \sum_{v \in L} \mu^{\times m}(v, e_S) e^{2\pi i \langle x, v \rangle} dx, \\ &= \int_{E/L^\star} \sum_{(v, s) \in A} \mu^{\times m}(v, s) e^{2\pi i \langle x, v \rangle} \varepsilon_{e_S}(s) dx. \end{aligned} \quad (4.11)$$

By equation (4.9) we have

$$(\rho_x(v, s) \varepsilon_{e_S})(e_S) = e^{2\pi i \langle x, v \rangle} \varepsilon_{e_S}(s).$$

Hence from equation (4.11) we have

$$\begin{aligned} \mu^{\times m}(0, e_S) &= \int_{E/L^\star} \sum_{(v, s) \in A} \mu^{\times m}(v, s) (\rho_x(v, s) \varepsilon_{e_S})(e_S) dx \\ &= \int_{E/L^\star} \sum_{a \in A} \mu^{\times m}(a) (\rho_x(a) \varepsilon_{e_S})(e_S) dx, \\ &= |S_{n+1}| \int_{E/L^\star} \langle \sum_{a \in A} \mu^{\times m}(a) \rho_x(a) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \end{aligned} \quad (4.12)$$

By equations (4.3) and (4.4), equation (4.12) becomes

$$\mu^{\times m}(0, e_S) = |S_{n+1}| \int_{E/L^\star} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx.$$

Let s be an element of S_{n+1} . Then, by Lemma 4.1.1, we obtain

$$\begin{aligned} \mu^{\times m}(0, e_S) &= |S_{n+1}| \int_{E/L^\star} \langle (\tilde{\mu}(\rho_{sx}))^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\ &= |S_{n+1}| \int_{E/L^\star} \langle (\tilde{\mu}(\rho_{sx}))^m \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx. \end{aligned}$$

By equation (4.4) and Lemma 4.1.1 we have

$$\begin{aligned}\rho_{sx}(\mu^{\times m}) &= \sum_{a \in A} \mu^{\times m}(a) \rho_{sx}(a), \\ &= \sum_{a \in A} \mu^{\times m}(a) L(s) \rho_x(a) L(s^{-1}).\end{aligned}$$

By Lemma 4.1.1, for all s in S ,

$$(\tilde{\mu}(\rho_{sx}))^m = \rho_{sx}(\mu^{\times m}) = L(s)(\tilde{\mu}(\rho_x))^m L(s^{-1}).$$

Hence

$$\begin{aligned}\mu^{\times m}(0, e_S) &= |S_{n+1}| \int_{E/L^*} \langle L(s)(\tilde{\mu}(\rho_x))^m L(s^{-1}) \varepsilon_{e_S}, \varepsilon_{e_S} \rangle dx, \\ &= |S_{n+1}| \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m L(s^{-1}) \varepsilon_{e_S}, L(s^{-1}) \varepsilon_{e_S} \rangle dx, \\ &= |S_{n+1}| \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m L(s) \varepsilon_{e_S}, L(s) \varepsilon_{e_S} \rangle dx, \\ &= |S_{n+1}| \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx.\end{aligned}$$

□

For the representation space $\mathbb{C}^{S_{n+1}}$, the set $\{\varepsilon_s : s \in S_{n+1}\}$ forms an orthogonal basis. Therefore, the trace of $(\tilde{\mu}(\rho_x))^m$ is given by

$$Tr \{ (\tilde{\mu}(\rho_x))^m \} = |S_{n+1}| \sum_{s \in S_{n+1}} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle.$$

Hence we have

$$\mu^{\times m}(0, e_S) = \frac{1}{|S_{n+1}|} \int_{E/L^*} Tr \{ (\tilde{\mu}(\rho_x))^m \} dx. \quad (4.13)$$

Similar to our property of the operator $P(x)$, x in E/L^* , whose eigenvalue are less or equal to 1 in absolute value, we have a useful criterion of the representation operator $\tilde{\mu}(\rho_x)$.

Lemma 4.2.2 *Let μ be a transition probability measure, and λ be an eigenvalue of $\tilde{\mu}(\rho_x)$, x inside a chamber. Then $|\lambda| < 1$. In other words, the norm $\| \tilde{\mu}(\rho_x) \|$ of the representation operator $\tilde{\mu}(\rho_x)$ is less than 1.*

Proof. We recall that

$$\|\tilde{\mu}(\rho_x)\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \tilde{\mu}(\rho_x)\},$$

since $\tilde{\mu}(\rho_x)$ is a self-adjoint operator (see equations (B.1) and (B.2) in Appendix B). Let λ be an eigenvalue of $\tilde{\mu}(\rho_x)$, x inside a chamber. Then for a corresponding eigenvector f we have

$$\lambda f = \tilde{\mu}(\rho_x)f. \quad (4.14)$$

By equation (4.4), we have

$$\begin{aligned} \lambda f(s) &= (\tilde{\mu}(\rho_x)f)(s), \\ &= \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s')(\rho_x(v',s')f)(s), \\ &= \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s')e^{2\pi i\langle x,sv' \rangle} f(ss'), \end{aligned} \quad (4.15)$$

for all s in S_{n+1} . Let s_0 be an element in S_{n+1} such that $|f(s_0)| = \max_{s \in S_{n+1}} |f(s)|$.

Then we find

$$\lambda f(s_0) = \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s')e^{2\pi i\langle x,s_0v' \rangle} f(s_0s'). \quad (4.16)$$

Since f is nonzero function, we have $f(s_0) \neq 0$ and hence from equation (4.16) we observe

$$\begin{aligned} |\lambda| &= \left| \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s')e^{2\pi i\langle x,s_0v' \rangle} \frac{f(s_0s')}{f(s_0)} \right|, \\ &\leq \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s') \left| e^{2\pi i\langle x,s_0v' \rangle} \frac{f(s_0s')}{f(s_0)} \right|, \\ &\leq 1. \end{aligned} \quad (4.17)$$

Hence each eigenvalue λ of $\tilde{\mu}(\rho_x)$ is less or equal to one in absolute value. Suppose there exist an eigenvalue λ of $\tilde{\mu}(\rho_x)$ with $\lambda = 1$. Then from equation (4.16) we find

$$1 \leq \sum_{(v',s') \in \text{Supp}(\mu)} \mu(v',s')e^{2\pi i\langle x,s_0v' \rangle} \frac{f(s_0s')}{f(s_0)}. \quad (4.18)$$

Suppose there exist an element (v', s') in $Supp(\mu)$ such that $|e^{2\pi i \langle x, s_0 v' \rangle} \frac{f(s_0 s')}{f(s_0)}| < 1$. Then inequality (4.18) leads to a contradiction. Hence we have

$$|e^{2\pi i \langle x, s_0 v' \rangle} \frac{f(s_0 s')}{f(s_0)}| = 1, \quad (4.19)$$

for all (v', s') in $Supp(\mu)$. By using the result in Appendix A we have

$$e^{2\pi i \langle x, s_0 v' \rangle} f(s_0 s') = f(s_0), \quad (4.20)$$

for all (v', s') in $Supp(\mu)$.

By Lemma (4.1.1) together with equation (4.14), we have

$$\begin{aligned} L(u)f &= L(u) \tilde{\mu}(\rho_x) L_{u^{-1}} L(u)f, \\ &= \rho_{ux}(\mu) L(u)f, \end{aligned}$$

for all u in S_{n+1} . This shows that $L(u)f$ is an eigenvector of $\rho_{ux}(\mu)$ with eigenvalue one. Hence by equation (4.20) we obtain

$$e^{2\pi i \langle ux, s_0 v \rangle} (L(u)s_0)(s_0 s) = (L(u)f)(s_0),$$

for all u in S and (v, s) in $Supp(\mu)$. By definition of left regular representation given in equation (4.1), we find

$$f(u^{-1} s_0 s) = e^{2\pi i \langle ux, s_0 v \rangle} f(u^{-1} s_0),$$

for all u in S_{n+1} , (v, s) in $Supp(\mu)$. Since the group A is generated by the set $Supp(\mu) = \{(v', s') \in A : \mu(v', s') \neq 0\}$ and A is isomorphic with the semidirect products of L and S_{n+1} , it follows that the set $G(S) = \{s' \in S_{n+1} : \mu(v', s') \neq 0 \text{ for some } v' \text{ in } L\}$ generates S_{n+1} . Hence letting $u^{-1} s_0 = w$, we get

$$|f(ws)| = |f(w)|,$$

for all w in S_{n+1} and s in $G(S)$. From definition of right regular representation R in equation (4.2), it immediately follows

$$(R(s)|f|)(w) = |f|(sw) = |f(sw)| = |f(w)| = |f|(w),$$

for all w in S_{n+1} and s in $G(S)$. In other words, $R(s)|f| = |f|$ for all s in $G(S)$. But for each s in S , there exists k in \mathbf{Z} such that $s = s_1 \dots s_k$ with s_1, \dots, s_k in $G(S)$. Hence for each s in $G(S)$ we obtain

$$R(s)|f| = R(s_1 \dots s_k)|f| = R(s_1) \dots R(s_k)|f| = |f|,$$

Hence for all s, u in S_{n+1} , $(R(s)|f|)(u) = |f|(u)$, and in particular,

$$|f|(s) = (R(s)|f|)(e) = |f|(e),$$

for all s in S_{n+1} . In other words, f is a constant function in absolute value. Hence $f(t) \neq 0$ for all t in S_{n+1} , since f is a nonzero function. Since f is an eigenvector of $\tilde{\mu}(\rho_x)$ with eigenvalue one, by equation (4.15) we obtain

$$\sum_{(v,s) \in A} \mu(v,s) e^{-2\pi i \langle x, tv \rangle} \frac{f(ts)}{f(t)} = 1,$$

for all t in S_{n+1} . By similar argument in obtaining (4.19), we have

$$|e^{-2\pi i \langle x, tv \rangle} \frac{f(ts)}{f(t)}| = 1,$$

for all $t \in S_{n+1}$. Hence, by Appendix A page 85, we find

$$e^{-2\pi i \langle x, tv \rangle} \frac{f(ts)}{f(t)} = 1,$$

for all $t, s \in S_{n+1}$ with $\mu(v, s) \neq 0$, for some $v \in L$, from which we obtain

$$\rho_x(v, s)f = f,$$

for all (v, s) in $Supp(\mu)$. Since $Supp(\mu)$ generates A , we have

$$\rho_x(a)f = f,$$

for all a in A . This implies that the subspace of $\mathbf{C}^{S_{n+1}}$ generated by f is a nontrivial subspace invariant under the action of $\rho_x(a)$, for all a in A which contradicts to the fact that ρ_x is irreducible. Therefore our assumption is false.

Using similar argument we can see that it is impossible for $\tilde{\mu}(\rho_x)$ to have a nonzero eigenvector f corresponding to the eigenvalue -1. (Also see Appendix B, Lemma B.0.13, page 87). As a result, we can conclude that $\|\tilde{\mu}(\rho_x)\| < 1$ for all x not in L^* .

□

Using this result, we close this section with the following theorem which is our generalization of Theorem 3.4.2.

Theorem 4.2.3 *The expected number of times the particle visits the origin is given by*

$$\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) = \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \left\{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \right\} dx.$$

where $\text{Tr} \left\{ (I - \tilde{\mu}(\rho_x))^{-1} \right\}$ denotes the trace of the operator $\left\{ (I - \tilde{\mu}(\rho_x))^{-1} \right\}$.

Proof. We note that for all x in E/L^* , the representation operator $\tilde{\mu}(\rho_x)$ is a bounded linear operator from the Hilbert space $\mathbf{C}^{S_{n+1}}$ into itself. Since the operator norm $\|\tilde{\mu}(\rho_x)\| \leq 1$, if $|\theta| < 1$, then $(I - \theta \tilde{\mu}(\rho_x))^{-1}$ exists as a bounded linear operator from $\mathbf{C}^{S_{n+1}}$ into itself and

$$(I - \theta \tilde{\mu}(\rho_x))^{-1} = \sum_{m=1}^{\infty} (\theta \tilde{\mu}(\rho_x))^m = I + \theta \tilde{\mu}(\rho_x) + \theta^2 (\tilde{\mu}(\rho_x))^2 + \dots$$

Since equation (4.2.1) in Lemma 4.2.1 holds for all s in S_{n+1} , we can change the factor $|S_{n+1}|$ in those equation for the sum over S_{n+1} . Hence the expected number of times the particle visits the origin is just the sum $\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S)$, and by monotone convergence theorem, from Lemma 4.2.1, we observe

$$\begin{aligned} \sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) &= \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \theta^m \mu^{\times m}(0, e_S), \\ &= \lim_{\theta \uparrow 1} \sum_{m=0}^{\infty} \sum_{s \in S_{n+1}} \int_{E/L^*} \theta^m \langle (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx, \end{aligned}$$

$$\begin{aligned}
&= \lim_{\theta \uparrow 1} \sum_{s \in S_{n+1}} \int_{E/L^*} \sum_{m=0}^{\infty} \langle \theta^m (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx, \\
&= \lim_{\theta \uparrow 1} \sum_{s \in S_{n+1}} \int_{E/L^*} \langle \sum_{m=0}^{\infty} \theta^m (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx, \\
&= \lim_{\theta \uparrow 1} \sum_{s \in S_{n+1}} \int_{E/L^*} \langle (I - \theta \tilde{\mu}(\rho_x))^{-1} \varepsilon_s, \varepsilon_s \rangle dx, \\
&= \lim_{\theta \uparrow 1} \int_{E/L^*} \sum_{s \in S_{n+1}} \langle (I - \theta \tilde{\mu}(\rho_x))^{-1} \varepsilon_s, \varepsilon_s \rangle dx, \\
&= \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \} dx.
\end{aligned}$$

□

It is clear that the formula in equation (4.13) is valid for any function f in $L^1(A)$ in place of μ . We can obtain a formula for $\mu^{\times m}(a)$ for any a in A using equation (4.13):

Note that $\mu^{\times m}(a) = (R(a)\mu^{\times m})(0, e_S)$, where R is the right regular representation.

Using (4.13) we set

$$\mu^{\times m}(a) = (R(a)(\mu^{\times m}))(0, e_S) = \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \{ \rho_x(R(a)\mu^{\times m}) \}.$$

By equations (4.2) and (4.4), we observe

$$\begin{aligned}
\rho_x(R(a)\mu^{\times m}) &= \sum_{b \in A} (R(a)\mu^{\times m})(b) \rho_x(b), \\
&= \sum_{b \in A} \mu^{\times m}(ba) \rho_x(b).
\end{aligned}$$

Letting $ba = c$,

$$\begin{aligned}
\rho_x(R(a)\mu^{\times m}) &= \sum_{c \in A} \mu^{\times m}(c) \rho_x(ca^{-1}), \\
&= \sum_{c \in A} \mu^{\times m}(c) \rho_x(c) \rho_x(a^{-1}), \\
&= \rho_x(\mu^{\times m}) \rho_x(a).
\end{aligned}$$

Hence we have

$$\mu^{\times m}(a) = \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \{ (\tilde{\mu}(\rho_x))^{\times m} \rho_x(a^{-1}) \} dx.$$

Therefore the expected number of times the particle visits the element a is given by

$$\sum_{i=0}^{\infty} \mu^{\times m}(a) = \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \sum_{m=0}^{\infty} \theta \text{Tr} \left((\tilde{\mu}(\rho_x))^{\times m} \rho_x(a^{-1}) \right) dx.$$

Using the fact that $\|\tilde{\mu}\| < 1$, we have

$$\sum_{i=0}^{\infty} \mu^{\times m}(a) = \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \left((I - \theta \tilde{\mu}(\rho_x))^{-1} \rho_x(a^{-1}) \right) dx.$$

Therefore, the conclusion of our results can be stated as follows.

Theorem 4.2.4 *Let μ be the transition probability measure. Then the random walk induced by μ is recurrent or transitory according as the integral*

$$\lim_{\theta \uparrow 1} \int_{E/L^*} \text{Tr} \{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \} dx \quad (4.21)$$

is infinite or finite respectively.

When we compare this result with the corresponding result in Chapter 3, we find that the ultimate behaviour for the symmetric random walk and the random walk with general probability measure are very similar which depends on the behaviour of the operators $(I - P(x))^{-1}$ and $(I - \tilde{\mu}(\rho_x))^{-1}$ respectively. In fact, it can be seen easily that $\tilde{\mu}(\rho_x) = P(x)$ when μ is considered to be the symmetric transition probability as in Chapter 3:

Let the probability measure μ of the random walk be defined as in Chapter 3.

Then the function μ can be written as

$$\mu = \frac{1}{n+1} \left\{ \sum_{i=1}^n \varepsilon_{(0, s_i)} + \varepsilon_{(r_0, s_0)} \right\},$$

where $\varepsilon_{(0, s_i)}$, $i = 1, \dots, n$, and $\varepsilon_{(r_0, s_0)}$ are indicator functions of the elements $(0, s_i)$, $i = 1, \dots, n$, and (r_0, s_0) of A .

By equations (4.3) and (4.4) we observe that, for all x not in L^* ,

$$\begin{aligned}
\tilde{\mu}(\rho_x) &= \rho_x(\mu), \\
&= \sum_{a \in A} \mu(a) \rho_x(a), \\
&= \frac{1}{n+1} \sum_{a \in A} \left(\sum_{i=1}^n \varepsilon_{(0, s_i)}(a) + \varepsilon_{(r_0, s_0)}(a) \right) \rho_x(a), \\
&= \frac{1}{n+1} \left(\sum_{i=1}^n \rho_x(0, s_i) + \rho_x(r_0, s_0) \right).
\end{aligned}$$

Hence for all x not in L^* , for all ψ in \mathbf{C}^S , we have

$$(\tilde{\mu}(\rho_x)\psi)(s) = \frac{1}{n+1} \left(\left[\sum_{i=1}^n \rho_x(0, s_i) + \rho_x(r_0, s_0) \right] \psi \right)(s).$$

Using equation (4.9) we obtain

$$\begin{aligned}
(\tilde{\mu}(\rho_x)\psi)(s) &= \frac{1}{n+1} \left(\sum_{i=1}^n \psi(ss_i) + e^{-2\pi i \langle x, sr_0 \rangle} \psi(ss_0) \right), \\
&= (P(x)\psi)(s),
\end{aligned}$$

for all s in S . Since this is true for all ψ in \mathbf{C}^S and for all s in S , it follows that $\tilde{\mu}(\rho_x) = P(x)$ for all x not in L^* . The integral given in Theorem 4.2.3 therefore becomes

$$\begin{aligned}
\lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \left\{ (I - \theta P(x))^{-1} \right\} dx &= \lim_{\theta \uparrow 1} \int_{E/L^*} \sum_{s \in S} \langle (I - \theta P(x))^{-1} \varepsilon_s, \varepsilon_s \rangle dx, \\
&= \lim_{\theta \uparrow 1} |S| \int_{E/L^*} \langle (I - \theta P(x))^{-1} \varepsilon_e, \varepsilon_e \rangle dx.
\end{aligned}$$

4.3 Random Movement

We can generalize further our discussion into a random movement on the collection of chambers \mathcal{C} . We shall show that our result in Theorem 4.2.3 remains valid and hence the ultimate behaviour of the movement is recurrent or transitory according as the integral in equation (4.21) is infinite or finite.

Suppose a particle starts at time $m = 0$ at the chamber C_0 and moves at times $m \geq 1$ to any chamber in \mathcal{C} with general transition probability μ so that the support $Supp(\mu)$ generates A . We also assume that the steps are statistically independent of the preceding steps. Hence in this case the particle at any time $m \geq 1$ can take either no step or a unit step to an adjacent chamber or even jump to another chamber according to the transition probability μ we defined.

Lemma 4.3.1 *The probability of the particle arriving at the origin after m steps is given by*

$$\mu^{\times m}(0, e_s) = |S_{n+1}| \int_{E/L^*} \langle (\tilde{\mu}(\rho_x))^m \varepsilon_s, \varepsilon_s \rangle dx,$$

for all s in S_{n+1} .

Proof. The reasoning we have given in proving Lemma 4.2.1 remains valid in this case. □

In evaluation of the expected number of times the particle visits the origin, we use the fact that the representation operator $\tilde{\mu}(\rho_x)$ is a self-adjoint operator. In our situation for random movement we do not enjoy this property, but fortunately we have the following lemma.

Lemma 4.3.2 *For all x not in L^* the norm $\|\tilde{\mu}(\rho_x)\|$ of the operator $\tilde{\mu}(\rho_x)$ is less or equal to one in absolute value.*

Proof. From equation (4.4), we observe that

$$\begin{aligned} \|\tilde{\mu}(\rho_x)\| &= \left\| \sum_{a \in Supp(\mu)} \mu(a) \rho_x(a) \right\|, \\ &\leq \sum_{a \in Supp(\mu)} \mu(a) \|\rho_x(a)\| = \sum_{a \in Supp(\mu)} \mu(a) = 1, \end{aligned} \quad (4.22)$$

since the representation operator $\rho_x(a)$ is unitary for all a in A .

□

Finally using this lemma, by similar argument in obtaining Theorem 4.2.3, we can prove the following theorem for the random movement.

Theorem 4.3.3 *The expected number of times the particle visits the origin is given by*

$$\sum_{m=0}^{\infty} \mu^{\times m}(0, e_S) = \lim_{\theta \uparrow 1} \frac{1}{|S_{n+1}|} \int_{E/L^*} \text{Tr} \left\{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \right\} dx.$$

where $\text{Tr} \left\{ (I - \tilde{\mu}(\rho_x))^{-1} \right\}$ denotes the trace of the operator $\left\{ (I - \tilde{\mu}(\rho_x))^{-1} \right\}$.

Hence we have the same conclusion with our previous discussion of random walk that the ultimate behaviour of the movement is recurrent or transitory according as the integral

$$\lim_{\theta \uparrow 1} \int_{E/L^*} \text{Tr} \left\{ (I - \theta \tilde{\mu}(\rho_x))^{-1} \right\} dx$$

is infinite or finite respectively.

Chapter 5

Intertwining Number Theorem of Locally Compact Groups

The main purpose of this chapter is to generalize Mackey's result on the intertwining number theorem for locally compact groups (see [16], Theorem 3', page 588).

Let G be a locally compact group and H, K be open and closed subgroups of G . Let π and γ be two one dimensional representations of H and K respectively. Then Mackey's Intertwining Number Theorem states that the intertwining number of the two induced representations U^π and U^γ of π and γ respectively, is equal to the sum of the intertwining numbers of the representations π^x and γ^y of $H^x \cap K^y$, where xy^{-1} runs through the set of all double coset representatives of H and K in G . Using Mackey's notations, the result can be stated by the formula

$$I(U^\pi, U^\gamma) = \sum_{D(x,y) \in \mathcal{D}_f} I(\pi, \gamma, D(x,y)),$$

where \mathcal{D}_f is the family of all double cosets $D(x,y) = Hxy^{-1}K$ for which the indices of $H^x \cap K^y$ in H^x and K^y are finite (see Section 5.2, page 77).

We intend to prove the fact that this result holds in the situation where the representations of the subgroups are finite dimensional. To achieve this, we will

use results in the theory of A_p^q spaces (see, for example, [2] and [23]). Recent developments in this theory suggest that, under certain conditions, the A_p^q spaces can be recognized as preduals of intertwining operators of induced representations. First, we state the simplified versions of these results under the condition that the subgroup are open and closed. Then we apply them to prove the generalized Mackey's Intertwining Number Theorem. We begin with some definitions and basic facts we need in our analysis.

5.1 Definitions and Basic Facts

In this section we review some definitions and basic facts of representations of locally compact groups, intertwining operators and tensor products. Some useful information may be found, for example, in [2], [5], [22], [26].

Let G be a locally compact group and H be an open and closed subgroup of G . A representation π of H on a Banach space $\mathcal{H}(\pi)$ is a continuous homomorphism π from the group H into the group $U(\mathcal{H}(\pi))$ of all isometries of $\mathcal{H}(\pi)$ onto itself. Let $(\mathcal{H}(\pi))^*$ denote the conjugate space of $\mathcal{H}(\pi)$. The map π^* from H to the space $U((\mathcal{H}(\pi))^*)$ of all isometries from $(\mathcal{H}(\pi))^*$ into itself given by

$$\pi^*(h) = (\pi(h^{-1}))^*,$$

is a representation of H on the Hilbert space $\mathcal{H}(\pi^*) = (\mathcal{H}(\pi))^*$ (see [2], page 38), where $(\pi(h^{-1}))^*$ is the adjoint operator of $\pi(h^{-1})$.

Let ρ_1 and ρ_2 be representations of G in Banach spaces $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_2)$ respectively. We define an intertwining operator τ for ρ_1 and ρ_2 as a bounded operator from $\mathcal{H}(\rho_1)$ into $\mathcal{H}(\rho_2)$ such that

$$\tau \rho_1(g) = \rho_2(g) \tau,$$

for all g in G . We denote $Hom_G(\mathcal{H}(\rho_1), \mathcal{H}(\rho_2))$ the space of all such intertwining

operators. The dimension of this space is called the intertwining number of the operators ρ_1 and ρ_2 and is denoted by $\mathcal{I}(\rho_1, \rho_2)$.

The definition of the p -induced representation U_p^π of G induced by a representation π (on a Banach space $\mathcal{H}(\pi)$) of a closed subgroup H can be found in many places in the literature (e.g. [2], [22], [23]).

Here, we consider the representation spaces of the subgroups to be Hilbert spaces and define the corresponding induced representations accordingly. Moreover, we only consider open and closed subgroups of G so that the homogeneous spaces obtained are denumerable and discrete.

Let H be an open and closed subgroup of G and π be a unitary representation of H on Hilbert space $\mathcal{H}(\pi)$. We consider the vector space of all functions f from G to $\mathcal{H}(\pi)$ satisfying the condition:

$$f(hg) = \pi(h)f(g),$$

for all h in H and g in G . Let us set $\|f\|^2 = \sum_{x \in G/H} \langle f(x), f(x) \rangle$. We define $L_2(\pi)$ to be the set of all functions f in the space under considerations for which $\|f\|$ is finite. Clearly, $L_2(\pi)$ is a Hilbert space under the norm just defined. The induced representation U^π of G is then defined by

$$(U^\pi(s)f)(x) = f(xs),$$

for all $s, x \in G$ and $f \in L_2(\pi)$. It is easy to show that $s \mapsto U^\pi(s)$ is a unitary representation of G .

Let π and γ be unitary representations of open and closed subgroups H and K of G respectively. Let T be a bounded linear operator from $L_2(\pi)$ into $L_2(\gamma)$. T is called an integral operator if there exists a summable function Φ , called the kernel of T , from $G \times G$ to $\mathcal{L}(\mathcal{H}(\pi), \mathcal{H}(\gamma))$ such that for a given f in $L_2(\pi)$,

1. the function $x \mapsto \Phi(y, x)f(x)$ is summable for all y in G/K ,

2. $y \mapsto \sum_{x \in G/H} \Phi(y, x)f(x)$ belongs to $L_2(\gamma)$, and

3. $(Tf)(y) = \sum_{x \in G/H} \Phi(y, x)f(x)$, for all y in G/K ,

(see [2], page 76). Note that we use the term “integral” only to be consistent with the case where the subgroups are not necessarily open and closed.

For x in G , let H^x be the subgroup of G consisting of all elements of the form $x^{-1}hx$, for all h in H . Then the map π^x from H^x into $U(\mathcal{H}(\pi))$ given by $\pi^x(b) = \pi(xbx^{-1})$ defines a representation of the group H^x . The properties of the kernel of an integral intertwining operator of induced representations U^π and U^γ is described in the following lemma.

Lemma 5.1.1 *Let Φ be the kernel of an integral intertwining operator of induced representations U^π and U^γ . Then the following statements hold.*

1. For all x in G/H , y in G/K , s in G , $h \in H$ and $k \in K$,

$$\Phi(kys, hxs)\pi(h) = \gamma(k)\Phi(y, x). \quad (5.1)$$

2. $\Phi(x, y)$ is an intertwining operator of the representations π^x and γ^y of the subgroup $H^x \cap K^y$ of G .

Proof. 1. (cf. [2], page 77). Let T be an integral operator from $L_2(\pi)$ to $L_2(\gamma)$ with the kernel Φ . Then for all f in $L_2(\pi)$ and y in G ,

$$(Tf)(y) = \sum_{x \in G/H} \Phi(y, x)f(x).$$

By hypothesis, T is in $\text{Hom}_G(L_2(\pi), L_2(\gamma))$. Hence

$$(TU^\pi(s)f)(y) = (U^\pi(s)Tf)(y), \quad (5.2)$$

for all y in G/K and s in G . Now observe that

$$\begin{aligned}(TU^\pi(s)f)(y) &= \sum_{x \in G/H} \Phi(y, x)(U^\pi(s)f)(x), \\ &= \sum_{x \in G/H} \Phi(y, x)f(xs).\end{aligned}$$

Changing variables $xs \mapsto x$, we find

$$(TU^\pi(s))(y) = \sum_{x \in G/H} \Phi(y, xs^{-1})f(x). \quad (5.3)$$

On the other hand,

$$\begin{aligned}(U^\gamma(s)Tf)(y) &= (Tf)(ys), \\ &= \sum_{x \in G/H} \Phi(ys, x)f(x).\end{aligned} \quad (5.4)$$

By equation (5.2) together with equations (5.3) and (5.4) we obtain, for all x in G/H , y in G/K and s in G ,

$$\Phi(y, xs^{-1}) = \Phi(ys, x). \quad (5.5)$$

Let $k \in K$ and $y \in G$. Let γ_k and π_h denote $\gamma(k)$ and $\pi(h)$ respectively. Then

$$\begin{aligned}\gamma_k(Tf)(y) &= (Tf)(ky) \\ &= \sum_{x \in G/H} \Phi(ky, x)f(x), \\ &= \sum_{x \in G/H} \Phi(ky, hx)\pi_h f(x),\end{aligned} \quad (5.6)$$

for all h in H . Also

$$\gamma_k(Tf)(y) = \gamma_k \sum_{x \in G/H} \Phi(y, x)f(x).$$

Hence, for all h in H , k in K , x in G/H and y in G/K , we have

$$\Phi(ky, hx)\pi_h = \gamma_k \Phi(y, x). \quad (5.7)$$

By equations (5.5) and (5.7) we observe

$$\Phi(kys, hxs)\pi_h = \Phi(ky, hx)\pi_h = \gamma_k\Phi(y, x). \quad (5.8)$$

2. We shall show that, $\gamma_b^y\Phi(y, x) = \Phi(y, x)\pi_b^x$, for all x in G/H , y in G/K and b in $H^x \cap K^y$. For this purpose, let $b = y^{-1}ky = x^{-1}hx$ be in $H^x \cap K^y$. Then using equation (5.7) we observe

$$\begin{aligned} \gamma_b^y\Phi(y, x) &= \gamma_{y^{-1}by}\Phi(y, x), \\ &= \Phi(yby^{-1}y, xgx^{-1}x)\pi_{xbx^{-1}}, \\ &= \Phi(yb, xb)\pi_b^x, \\ &= \Phi(x, y)\pi_b. \end{aligned}$$

Since this is true for all x in G/H , y in G/K and b in $H^x \cap K^y$, (2) follows. \square

5.1.1 G -tensor Products of Banach Spaces

Let V, X_1, X_2 be vector spaces. An operator B from $X_1 \times X_2$ to V is said to be a bilinear operator if for all x, y in $X_1 \times X_2$ the maps B_x from X_2 to V given by $B_x(y) = B(x, y)$ and B_y from X_1 to V given by $B_y(x) = B(x, y)$ are linear operators. In other words, B is bilinear if it is linear in each of the variables x and y .

A tensor product of X_1 and X_2 is a pair consisting of a vector space W and a bilinear operator B from $X_1 \times X_2$ into W so that whenever (v_{1i}) is a basis of X_1 and (v_{2j}) is a basis of X_2 , the set of elements $B(v_{1i}, v_{2i})$ forms a basis for W . Since the existence of the space W is unique, we write W as the space $X_1 \otimes X_2$ and its element $x_1 \otimes x_2$, x_1 in X_1 and x_2 in X_2 .

If X_1 and X_2 are Banach spaces, then it is possible to endow norms in $X_1 \otimes X_2$. Let σ be a norm in $X_1 \otimes X_2$ defined by

$$\sigma(z) = \inf_{\substack{x_i \in X_1 \\ y_i \in X_2}} \sum_{i=1}^n \|x_i\| \|y_i\|,$$

for all $z = \sum_{i=1}^n x_i \otimes y_i$, where the infimum is taken over all possible representation of z . The norm σ is called the greatest-cross norm. Then the completion of $X_1 \otimes X_2$ with respect to the norm σ is a Banach space and denoted by $X_1 \otimes^\sigma X_2$. The Banach space $X_1 \otimes^\sigma X_2$ is called the projective tensor product of Banach spaces X_1 and X_2 .

Let G be a locally compact group and ρ_1 and ρ_2 representations of G in Banach spaces $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_2)$ respectively. A bounded bilinear function B from $\mathcal{H}(\rho_1) \times \mathcal{H}(\rho_2)$ into a Banach space X is called G -balanced if

$$B(\rho_1(g)v, w) = B(v, \rho_2(g)w),$$

for all g in G and (v, w) in $\mathcal{H}_1 \times \mathcal{H}_2$.

A G -tensor product of the Banach spaces $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_2)$ is a pair (\mathcal{H}, B_0) consisting a Banach space \mathcal{H} and bounded G -balanced bilinear function B_0 from $\mathcal{H}(\rho_1) \times \mathcal{H}(\rho_2)$ into \mathcal{H} , whose range spans \mathcal{H} , such that (\mathcal{H}, B_0) has universality property: for every bounded G -balanced bilinear operator B from $\mathcal{H}(\rho_1) \times \mathcal{H}(\rho_2)$ to a Banach space X , there is (necessarily unique) bounded linear operator T_B from \mathcal{H} to X such that $B = T_B B_0$. The operator T_B is said to be the operator associated with B .

For any two representations ρ_1 and ρ_2 of G in Banach space $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_2)$ respectively we can construct G -tensor product of $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_2)$. Let L be the closed linear subspace of $\mathcal{H}(\rho_1) \otimes^\sigma \mathcal{H}(\rho_2)$ spanned by the elements of the form

$$\rho_1(g)x_1 \otimes x_2 - x_1 \otimes \rho_2(g)x_2,$$

for all g in G and for all (x_1, x_2) in $\mathcal{H}_1 \times \mathcal{H}_2$. Then the quotient Banach space $(\mathcal{H}_1 \otimes^\sigma \mathcal{H}_2)/L$ is a G -tensor product of \mathcal{H}_1 and \mathcal{H}_2 and is denoted by $\mathcal{H}_1 \otimes_G^\sigma \mathcal{H}_2$

5.2 A_2^2 Space

The purpose of this section is to examine a particular A_2^2 space arising from spaces of induced representations U^π and U^γ of a locally compact group G , where π and γ are unitary representations of open and closed subgroups H and K of G respectively. Recent developments in the study of A_p^q spaces, with some conditions on p and q , can be found in [2] from which we quote the definitions and results.

Let G be a locally compact group and H and K be both open and closed subgroups of G . Let π and γ be finite dimensional representation of H and K respectively on Hilbert spaces $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma)$ respectively. Let U^π and U^γ be the corresponding induced representations of G of representations π and γ respectively. Then there exists a natural isometric isomorphism from G -tensor product $(L_2(\pi) \otimes_G^\sigma L_2(\gamma^*))$ of induced representation spaces $L_2(\pi)$ and $L_2(\gamma^*)$ onto the Hilbert space $Hom_G(L_2(\pi), L_2(\gamma))$ of all their intertwining operators (see [22]). In other words

$$(L_2(\pi) \otimes_G^\sigma L_2(\gamma^*))^* \cong Hom_G(L_2(\pi), L_2(\gamma)). \quad (5.9)$$

Consider the projective tensor product $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)$. For x and y in G , let $\mathcal{H}_{x,y}$ be the closed subspace of $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)$ spanned by the elements of the form

$$\pi^x(b)\xi \otimes \eta - \xi \otimes (\gamma^y(b))^*\eta,$$

for all b in $H^x \cap K^y$, ξ in $\mathcal{H}(\pi)$ and η in $\mathcal{H}(\gamma^*)$. The quotient Banach space $(\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*))/\mathcal{H}_{x,y}$ is denoted by $\mathcal{A}_{x,y}$. We note that $\mathcal{A}_{x,y}$ is $(H^x \cap K^y)$ -tensor product of $\mathcal{H}(\pi)$ and $\mathcal{H}(\gamma^*)$. Hence we have $\mathcal{A}_{x,y} = \mathcal{H}(\pi) \otimes_{(H^x \cap K^y)}^\sigma \mathcal{H}(\gamma^*)$. It can be seen easily (see [2], page 58) that for all s in G we have

$$\mathcal{H}_{sx, sy} = \mathcal{H}_{x,y} \quad \text{and} \quad \mathcal{A}_{xs, ys} = \mathcal{A}_{x,y}.$$

For $u \otimes v$ in $\mathcal{H}(\pi) \otimes^\sigma \mathcal{H}(\gamma^*)$ we write $u \otimes_{x,y} v$ to denote element of $\mathcal{A}_{x,y}$ to

which $u \otimes v$ belongs.

Lemma 5.2.1 *Let x and y be elements in G/H and G/K respectively. Then the following statements hold.*

$$\sum_{i=1}^{\infty} f_i \otimes g_i \in L_2(\pi) \otimes^{\sigma} L_2(\gamma^*) \quad \text{implies} \quad \sum_{i=1}^{\infty} f_i(x) \otimes g_i(y) \in \mathcal{H}(\pi) \otimes^{\sigma} \mathcal{H}(\gamma^*).$$

Proof. See [2], Proposition 4.1.2 page 56. □

Lemma 5.2.2 *For all x, y in G and $\sum_{i=1}^{\infty} f_i \otimes g_i \in L_2(\pi) \otimes^{\sigma} L_2(\gamma^*)$,*

$$t \mapsto \sum_{i=1}^{\infty} f_i(xt) \otimes_{x,y} g_i(yt),$$

is a mapping on the coset space $G/(H^x \cap K^y)$.

Proof. (cf. [2] Proposition 4.1.5 page 59). Let s be an element in $H^x \cap K^y$. We observe that

$$\begin{aligned} \sum_{i=1}^{\infty} f_i(xst) \otimes_{x,y} g_i(yst) &= \sum_{i=1}^{\infty} f_i(xsx^{-1}xt) \otimes_{x,y} g_i(ysy^{-1}yt), \\ &= \sum_{i=1}^{\infty} \pi^x(s) f_i(xt) \otimes_{x,y} \gamma^{*y}(s) g_i(yt), \\ &= \sum_{i=1}^{\infty} f_i(xt) \otimes_{x,y} g_i(yt). \end{aligned}$$
□

Let $\nabla = \{(x, x) : x \in G \times G\}$ be the diagonal subgroup of $G \times G$. Let \mathcal{D} be the family of all double cosets $(H \times K) : \nabla$ and $D(x, y)$ the coset in \mathcal{D} to which (x, y) belongs.

Lemma 5.2.3 *Let $\sum_{i=1}^{\infty} f_i \otimes g_i$ be an element in $L_2(\pi) \otimes^{\sigma} L_2(\gamma^*)$. For x, y in G , if the indices of $H^x \cap K^y$ in H^x and K^y are both finite, then the sum*

$$\sum_{i=1}^{\infty} f_i(xt) \otimes g_i(yt),$$

is finite for all $D(x, y)$ in \mathcal{D} .

Proof. (cf. [2], page 65). Using Cauchy Schwarz inequality we observe that

$$\sum_{t \in G/(H^x \cap K^y)} \sum_{i=1}^{\infty} \|f_i(xt)\| \|g_i(yt)\| \leq \sum_{i=1}^{\infty} \left(\sum_{t \in G/(H^x \cap K^y)} \|f_i(xt)\|^2 \right)^{\frac{1}{2}} \left(\sum_{t \in G/(H^x \cap K^y)} \|g_i(yt)\|^2 \right)^{\frac{1}{2}}.$$

But

$$\begin{aligned} \sum_{t \in G/(H^x \cap K^y)} \|f_i(xt)\|^2 &= \sum_{t \in G/H^x} \sum_{b \in H^x/(H^x \cap K^y)} \|f_i(xbt)\|^2, \\ &= \sum_{t \in G/H^x} \sum_{b \in H^x/(H^x \cap K^y)} \|f_i(xt)\|^2, \\ &= \sum_{b \in H^x/(H^x \cap K^y)} \sum_{t \in G/H^x} \|f_i(xt)\|^2, \\ &= \sum_{b \in H^x/(H^x \cap K^y)} \sum_{t \in G/H} \|f_i(t)\|^2, \\ &\leq M \|f_i\|_2, \end{aligned}$$

for some constant M , since the indices of $H^x \cap K^y$ in H^x and K^y are both finite. Similarly, we have $\sum_{t \in G/(H^x \cap K^y)} \|g_i(xt)\|^2 \leq N \|g_i\|_2$, for some constant N . Hence we obtain

$$\sum_{i=1}^{\infty} f_i(xt) \otimes g_i(yt) \leq S \sum_{i=1}^{\infty} \|f_i\|_2 \|g_i\|_2 < \infty,$$

for some constant $S = MN$, since $\sum_{i=1}^{\infty} f_i \otimes g_i$ is in $L_2(\pi) \otimes_G^{\sigma} L_2(\gamma^*)$.

□

In view of lemmas 5.2.2 and 5.2.3 we have the following definition (see [2], Definition 4.1.7 page 66).

Definition 5.2.4 Suppose the indices of $H^x \cap K^y$ in H^x and K^y are finite for all x, y in G . The map Ψ on $L_2(\pi) \otimes_G^{\sigma} L_2(\gamma^*)$ is defined by

$$\left(\Psi \left(\sum_{i=1}^{\infty} f_i \otimes g_i \right) \right) (x, y) = \sum_{G/(H^x \cap K^y)} \sum_{i=1}^{\infty} f_i(xt) \otimes_{x,y} g_i(yt),$$

for all $\sum_{i=1}^{\infty} f_i \otimes g_i$ in $L_2(\pi) \otimes_G^{\sigma} L_2(\gamma^*)$.

The value of $(\Psi(\sum_{i=1}^{\infty} f_i \otimes g_i))(x, y)$ for all x, y in G , belongs to the quotient space $\mathcal{A}_{x,y}$. The properties of this image space are being discussed in [2] (page 70).

Definition 5.2.5 *The space A_2^2 is defined to be the range of Ψ with quotient norm (see [2], page 72).*

The generalization of Rieffel's result on classical A_p^q spaces, (see [23]), is given in [2] (page 88, Theorem 4.4.3). It states, in our context, the following.

Theorem 5.2.6 *Suppose the indices $H^x \cap K^y$ in H^x and K^y are finite for all x, y in G . If the elements of $\text{Hom}_G(L_2(\pi), L_2(\gamma))$ are integral operators, then*

$$L_2(\pi) \otimes_G^\sigma L_2(\gamma^*) \cong A_2^2.$$

Proof. See [2], Theorem 4.4.7 page (88).

□

In the case where the subgroups H and K are open and closed subgroups, we have the following result given by Mackey [16].

Theorem 5.2.7 *Let π and γ be representations of the open and closed subgroups H and K of a locally compact group G . Then an intertwining operator of the induced representations U^π and U^γ of G is an integral operator. Furthermore, the corresponding kernel Φ satisfies the following conditions.*

1. $\Phi(kys, hxs) = \gamma_k \Phi(y, x) \pi_h^*$, for all $h \in H, k \in K, x, y, s \in G$.
2. $\sum_{x \in G/H} \frac{\|\Phi(y, x)\|^2}{\|v\|^2} \leq K$, for all $y \in G$ and $v \in \mathcal{H}(\pi)$, for some positive constant K .
3. $\sum_{y \in G/K} \frac{\|\Phi(y, x)^*\|^2}{\|v\|^2} \leq K'$, for all $x \in G$ and $v \in \mathcal{H}(\gamma)$, for some positive constant K' .
4. For all $f \in L_2(\pi)$, $(Tf)(x) = \sum_{x \in G/H} \Phi(y, x) f(x)$.

5. For all $f \in L_2(\gamma)$, $(Tf)(x) = \sum_{y \in G/K} \Phi(y, x)^* f(y)$.

Proof. See [16], Lemma A, page 585. □

Combining two theorems 5.2.6 and 5.2.7 we have

Theorem 5.2.8 *Let H and K be open and closed subgroup of a locally compact group G . Let π and γ be representations of H and K and $L_2(\pi)$ and $L_2(\gamma)$ the corresponding induced representation spaces of G . If the indices of $H^x \cap K^y$ in H^x and K^y are finite, then*

$$L_2(\pi) \otimes_G^\sigma L_2(\gamma^*) \cong A_2^2.$$

Proof. This is a straight forward consequence of Theorem 5.2.6 and Theorem 5.2.7. □

The diagonal subgroup ∇ acts on the coset space $(G \times G)/(H \times K)$ on the right; and the stabilizer of the coset (Hx, Ky) under this action is $(H \times K)^{(x,y)} \cap \nabla$. The orbit is the double coset $(H \times K)(x, y)\nabla$. It is clear that $(H \times K)^{(x,y)} \cap \nabla$ can be identified with $H^x \cap K^y$. Moreover, (x_0, y_0) and (x_1, y_1) belong to the same $(H \times K) : \nabla$ double coset if and only if $x_0 y_0^{-1}$ and $x_1 y_1^{-1}$ belong to the same $H : K$ double coset.

Let \mathcal{D}_f be the set of all double cosets $D(x, y)$ for which the indices of $H^x \cap K^y$ in H^x and K^y are both finite.

Theorem 5.2.9 *Let π and γ be finite dimensional representations of the open and closed subgroups H and K of G . If Φ is the kernel of a bounded intertwining operator and if $\Phi(y, x) \neq 0$ for some x, y , then the double coset $D(x, y)$ is in \mathcal{D}_f .*

Proof. For all x, y in G , $\Phi(y, x) \in \text{Hom}_{H^x \cap K^y}(\mathcal{H}(\pi^x), \mathcal{H}(\gamma^y))$ by Theorem 5.2.7

(3). Hence there exists $\Theta(y, x)$ in $(\mathcal{H}(\pi^x) \otimes_{H^x \cap K^y} \mathcal{H}(\gamma^y))^*$ such that

$$\langle v \otimes_{x,y} w, \Theta(y, x) \rangle = \langle w, \Phi(y, x)v \rangle, \quad (5.10)$$

for all $v \in \mathcal{H}(\pi^x)$ and $w \in (\mathcal{H}(\gamma^y))^*$. We observe now,

$$\begin{aligned} \sum_{x \in G/H} |\langle v \otimes_{x,y} w, \Theta(y, x) \rangle|^2 &= \sum_{x \in G/H} |\langle w, \Phi(y, x)v \rangle|^2, \\ &\leq \sum_{x \in G/H} \|w\|^2 \|\Phi(y, x)v\|^2, \\ &\leq \|w\|^2 \|T\|^2 \|v\|^2, \end{aligned} \quad (5.11)$$

by Theorem 5.2.7 (2). Let $\{l_i\}$ be a set of right coset representatives of $H^x \cap K^y$ in K^y . Then the double coset $Hxy^{-1}K$ is a disjoint union of cosets $Hx l_i y^{-1}$. From equation (5.10), we observe that

$$\begin{aligned} \langle w \otimes_{e, x l_i y^{-1}} v, \Theta(e, x l_i y^{-1}) \rangle &= \langle w, \Phi(e, x l_i y^{-1})v \rangle, \\ &= \langle w, \Phi(y l_i^{-1} y^{-1} y, x)v \rangle \quad \text{by (5.1),} \\ &= \langle w, \gamma_{l_i}^y \Phi(y, x)v \rangle \quad \text{by (5.1),} \\ &= \langle (\gamma_{l_i}^y)^*(w), \Phi(y, x)v \rangle, \\ &= \langle v \otimes_{x,y} (\gamma_{l_i}^y)^*(w), \Theta(y, x) \rangle. \end{aligned} \quad (5.12)$$

If π and γ are finite dimensional and the index of $H^x \cap K^y$ in K^y is infinite, then the set $\{l_i\}$ is infinite. Let $\{r_1, \dots, r_m\}$ be an orthonormal basis in the space $\mathcal{H}(\pi^x) \otimes \mathcal{H}(\gamma^y)^*$. Then the unit ball in $\mathcal{H}(\pi^x) \otimes \mathcal{H}(\gamma^y)^*$ is compact and hence there exists a subsequence $\{l_{i_n}\}$ such that

$$(\pi_e \otimes (\gamma_{l_{i_n}}^y)^*)(r_j)$$

converges to some r'_j for each $j = 1, \dots, m$. Note that $\{r'_1, \dots, r'_m\}$ forms an orthonormal basis for $\mathcal{H}(\pi^x) \otimes \mathcal{H}(\gamma^y)^*$; and since $\Theta(y, x) \neq 0$, there exists r'_j

such that r'_j is not in the Null space of $\Theta(y, x)$. Hence there exists some positive number p such that $\langle r'_j, \Theta(y, x) \rangle > p$. Also, there exists positive number q such that

$$\langle r, \Theta(y, x) \rangle > p \text{ whenever } \|r - r'_j\| < q. \quad (5.13)$$

Since $(\pi_e \otimes (\gamma_{l_{i_n}}^y)^*)(r_j)$ converges to r'_j , it follows that

$$\|(\pi_e \otimes (\gamma_{l_{i_n}}^y)^*)(r_j) - r'_j\| < q,$$

for large n . By equation (5.13), we have $\langle (\pi_e \otimes \gamma_{l_{i_n}}^y)(r_j), \Theta(y, x) \rangle > p$, for large n . This contradicts the finiteness of (5.11). Therefore the set $\{l_i\}$ must be finite. Using a similar argument together with conditions 3 in Theorem 5.2.7 we find that $[H^x : H^x \cap K^x] < \infty$.

□

Theorem 5.2.10 *Let H and K be open and closed subgroup of a locally compact group G . Let π and γ be finite dimensional representations of H and K and $L_2(\pi)$ and $L_2(\gamma)$ the corresponding induced representation spaces of G . Then we have*

$$L_2(\pi) \otimes_G^\sigma L_2(\gamma^*) \cong A_2^2.$$

Proof. This is a direct consequence of Theorem 5.2.8 and Theorem 5.2.9.

□

The intertwining number of the representations π^x and γ^y of $H^x \cap K^y$ only depends on the double cosets $D = D(x, y) = Hxy^{-1}K$ to which xy^{-1} belongs (see [16], Theorem 3'). Hence the intertwining number of π^x and γ^y can be denoted by $\mathcal{I}(\pi, \gamma, D)$. Generalizing Mackey's Intertwining Number Theorem (see [16] Theorem 3') we have the following result.

Theorem 5.2.11 *Let H and K be open and closed subgroup of a locally compact group G . Let π and γ be representations of H and K and $L_2(\pi)$ and $L_2(\gamma)$ the corresponding induced representation spaces of G . Then we have*

$$\mathcal{I}(U^\pi, U^\gamma) = \sum_{D \in \mathcal{D}_f} \mathcal{I}(\pi, \gamma, D).$$

Proof. Let F be a bounded linear functional on $L_2(\pi) \otimes_G^\sigma L_2(\gamma^*)$. Then there corresponds an operator $T \in \text{Hom}_G(L_2(\pi), L_2(\gamma))$ such that

$$\langle r, F \rangle = \sum_{i=1}^{\infty} \langle g_i, T f_i \rangle, \quad (5.14)$$

for any r in $L_2(\pi) \otimes_G^\sigma L_2(\gamma^*)$ with the expansion $r = \sum_{i=1}^{\infty} f_i \otimes g_i$. Using the discussion preceding the Theorem 5.2.9 we observe that

$$\begin{aligned} \sum_{i=1}^{\infty} \langle g_i, T f_i \rangle &= \sum_{i=1}^{\infty} \sum_{y \in G/K} \langle g_i(y), T f_i(y) \rangle, \\ &= \sum_{i=1}^{\infty} \sum_{y \in G/K} \sum_{x \in G/H} \langle g_i(y), \Phi(y, x) f(x) \rangle, \\ &= \sum_{i=1}^{\infty} \sum_{(x, y) \in (G \times G)/(H \times K)} \langle g_i(y), \Phi(y, x) f(x) \rangle, \\ &= \sum_{i=1}^{\infty} \sum_{D(x, y) \in \mathcal{D}} \sum_{t \in G/(H^x \cap K^y)} \langle g_i(yt), \Phi(y, x) f_i(xt) \rangle, \\ &= \sum_{i=1}^{\infty} \sum_{D \in \mathcal{D}} \sum_{t \in G/(H^x \cap K^y)} \langle g_i(yt), \Phi(y, x) f_i(xt) \rangle, \\ &= \sum_{i=1}^{\infty} \sum_{D \in \mathcal{D}} \sum_{t \in G/(H^x \cap K^y)} \langle f_i(xt) \otimes_{x, y} g_i(yt), \Theta(y, x) \rangle, \\ &= \sum_{D \in \mathcal{D}} \left\langle \left(\Psi \left(\sum_{i=1}^{\infty} f_i \otimes g_i \right) \right) (x, y), \Theta(y, x) \right\rangle, \\ &= \left\langle \Psi \left(\sum_{i=1}^{\infty} f_i \otimes g_i \right), \Theta \right\rangle. \end{aligned} \quad (5.15)$$

Using (5.9) and Theorem 5.2.10, we obtain

$$(A_2^2)^* \cong \text{Hom}_G(L_2(\pi), L_2(\gamma)). \quad (5.16)$$

By equations (5.15) and (5.16), the intertwining number $\mathcal{I}(U^\pi, U^\gamma)$ is equal to the dimension of all functions Θ which is equal to the dimension of the space of all functions Φ . Since the value of Φ is simply determined by its value $\Phi(x_0, y_0)$ at $(x_0, y_0) \in D$, we have

$$\mathcal{I}(U^\pi, U^\gamma) = \sum_{D \in \mathcal{D}} d_D,$$

where d_D is the dimension of all functions Φ which vanish outside the double coset D . Using Theorem 5.2.10, we have

$$\mathcal{I}(U^\pi, U^\gamma) = \sum_{D \in \mathcal{D}_f} \mathcal{I}(\pi, \gamma, D).$$

□

Appendix A

A Fact from Complex Numbers

In this thesis we frequently use the following fact.

Lemma A.0.12 *Let $\{\alpha_n\}_{n=1}^N$ be a set of positive real numbers with $\sum_{n=1}^N \alpha_n = \alpha_0$. Suppose we have*

$$\sum_{n=1}^N \alpha_n e^{i\theta_n} = \alpha_0,$$

where θ_n is real number depending on n . Then each $e^{i\theta_n} = 1$, for all $n = 1, \dots, N$.

Proof. We use induction on N . For $N = 1$, it is obvious that the statement is trivial. Let us assume that the statement is true for some positive integer $N - 1$.

Then we have

$$\alpha_0 = \left| \sum_{n=1}^N \alpha_n e^{i\theta_n} \right| \leq \left| \sum_{n=1}^{N-1} \alpha_n e^{i\theta_n} \right| + \alpha_N = \alpha_0. \quad (\text{A.1})$$

Hence we have equality in (A.1). This implies that

$$\left| \sum_{n=1}^{N-1} \alpha_n e^{i\theta_n} \right| = \alpha_0 - \alpha_N = |\alpha_0 - \alpha_N|. \quad (\text{A.2})$$

On the other hand,

$$\left| \sum_{n=1}^{N-1} \alpha_n e^{i\theta_n} \right| = \left| \sum_{n=1}^{N-1} \alpha_n e^{i\theta_n} - \alpha_N e^{i\theta_N} \right| = |\alpha_0 - \alpha_N e^{i\theta_N}|. \quad (\text{A.3})$$

From equations (A.2) and (A.3), we obtain

$$|\alpha_0 - \alpha_N| = |\alpha_0 - \alpha_N e^{i\theta_N}|,$$

which implies that $e^{i\theta_N} = 1$. Hence equation (A.3) becomes

$$\sum_{n=1}^{N-1} \alpha_n e^{i\theta_n} = \alpha_0 - \alpha_N \quad (\text{A.4})$$

Using induction assumption, equation (A.4) together with the fact that $\sum_{n=1}^{N-1} \alpha_n = \alpha_0 - \alpha_N$ gives the result that $e^{i\theta_n} = 1$, for each $n = 1, \dots, N-1$. Thus we have $e^{i\theta_n} = 1$, for all $n = 1, \dots, N$. Therefore, by Principle of Mathematical Induction, the statement is true for all N in \mathbb{N} .

Alternatively, we can prove the above as follows. We observe that

$$\begin{aligned} \sum_{n=1}^N |\sqrt{\alpha_n} e^{i\theta_n} - \sqrt{\alpha_n}|^2 &= \sum_{n=1}^N \left(\alpha_n |e^{i\theta_n}|^2 + \alpha_n - 2\operatorname{Re} [\alpha_n e^{i\theta_n}] \right), \\ &= 2 \sum_{n=1}^N \alpha_n - 2\operatorname{Re} \left[\sum_{n=1}^N \alpha_n e^{i\theta_n} \right], \\ &= 0. \end{aligned}$$

Hence $\sqrt{\alpha_n} e^{i\theta_n} - \sqrt{\alpha_n} = 0$ for all $n = 1, \dots, N$ from which we obtain $e^{i\theta_n} = 1$ for all $n = 1, \dots, N$.

□

Appendix B

Some Further Properties of $\tilde{\mu}(\rho_x)$

In Chapter 3 we have some properties concerning the operator $P(x)$ for all x in E/L^* as given in Lemma 3.3.2. For a general random walk with probability measure μ , the following Lemma is a generalization of those lemma.

Lemma B.0.13 *Let x be an element of E/L^* , μ the probability measure and λ an eigenvalue of $\tilde{\mu}(\rho_x)$. Then the following statements holds.*

1. $|\lambda| \leq 1$.
2. $\tilde{\mu}(\rho_0)$ has 1 as a simple eigenvalue with all nonzero constant functions as the corresponding eigenvectors.
3. If $x \neq 0$, then $(I - \tilde{\mu}(\rho_x))$ is invertible.

Proof. 1. This is proved in Chapter 4 Lemma 4.2.2. Alternatively, we can prove (1) simply as follows. From equation (4.22) page 67, we have $\|\tilde{\mu}(\rho_x)\| \leq 1$. In addition, it can be seen easily that the linear operator $\tilde{\mu}(\rho_x)$ is a self adjoint operator on a complex Hilbert space: In fact, for all f in \mathbb{C}^S , we have

$$(\rho_x(0, s_i)f)(s) = f(ss_i) = (S_i f)(s), \quad (\text{B.1})$$

for all $i = 1, \dots, n$, and

$$(\rho_x(r_0, s_0)f)(s) = e^{-2\pi i \langle x, s_0 r_0 \rangle} f(ss_0) = (K(x)f)(s), \quad (\text{B.2})$$

where S_i and $K(x)$ are linear operators defined in Chapter 3. Hence all eigenvalues λ of $\tilde{\mu}(\rho_x)$ are less or equal to 1 in absolute value.

2. By equations (4.9) and (4.4) we observe that for all f in $L^2(A/L, \chi_0)$ and (v, s) in A , we have

$$(\tilde{\mu}(\rho_0)f)(v, s) = \sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0) f(ss_0). \quad (\text{B.3})$$

Hence if f is a constant function, then for all (v, s) in A ,

$$(\tilde{\mu}(\rho_0)f)(v, s) = f(ss_0) \sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0) = f(ss_0) = f(v, s).$$

Hence $\tilde{\mu}(\rho_0)$ has the eigenvalue 1 with all nonzero constant functions as the corresponding eigenvectors. Now let f be an eigenvector of $\tilde{\mu}(\rho_0)$ with the corresponding eigenvalue 1. Then for all (v, s) in A , we have

$$(\tilde{\mu}(\rho_0)f)(v, s) = f(v, s).$$

Hence by equation (B.3), we obtain

$$\begin{aligned} (\tilde{\mu}(\rho_0)f)(s) &= \sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0) f(ss_0), \\ &= f(s). \end{aligned}$$

Let t be an element of S_{n+1} such that $|f(t)| = \max\{|f(s)| : s \in S_{n+1}\}$. Then, for all (v_0, s_0) in $\text{Supp}(\mu)$, since f is nonzero function, we have

$$\sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0) \frac{f(ts_0)}{f(t)} = 1. \quad (\text{B.4})$$

Suppose there exists even only one (v_0, s_0) in $Supp(\mu)$ with $|\frac{f(ts_0)}{f(t)}| < 1$. Then equation (B.4) becomes

$$\begin{aligned} 1 &= \left| \sum_{(v_0, s_0) \in Supp(\mu)} \mu(v_0, s_0) \frac{f(ts_0)}{f(t)} \right|, \\ &\leq \sum_{(v_0, s_0) \in Supp(\mu)} \mu(v_0, s_0) \left| \frac{f(ts_0)}{f(t)} \right|, \\ &< 1, \end{aligned}$$

which is a contradiction. Hence we have

$$\left| \frac{f(ts_0)}{f(t)} \right| = 1, \quad (\text{B.5})$$

for all (v_0, s_0) in $Supp(\mu)$.

Hence from equations (B.4), (B.5) and the fact that $\sum_{(v_0, s_0) \in Supp(\mu)} \mu(v_0, s_0)$, by Appendix A, we obtain

$$f(ts_0) = f(t),$$

for all $(v_0, s_0) \in Supp(\mu)$. Since the group A is generated by the set $Supp(\mu) = \{(v_0, s_0) \in A : \mu(v_0, s_0) \neq 0\}$ and A is isomorphic to the semidirect product of L and S_{n+1} , it follows that S_{n+1} is generated by the set $\{s \in S_{n+1} : \mu(v, s) \neq 0, \text{ for some } v \text{ in } L\}$. Hence we have

$$f(ts_0) = f(t),$$

for all s_0 in S_{n+1} . Using the fact that $(R(s_0)f)(t) = f(t)$, for all s_0 in S_{n+1} and $f(v, s) = e^{-2\pi i \langle 0, v \rangle} f(s) = f(s)$, for all (v, s) in A , we find that f is a nonzero constant function. Therefore, the eigenspace consisting of constant functions corresponding to the eigenvalue 1 is of dimension one, since it is generated by the single element 1_A .

3. $(I - \tilde{\mu}(\rho_x))$ is invertible if and only if $\lambda = 1$ is not an eigenvalue of $\tilde{\mu}(\rho_x)$. Let $\lambda = 1$ be an eigenvalue of $\tilde{\mu}(\rho_x)$. Then by equations (4.9) and (4.4), for a

corresponding eigenvector f , we have

$$\begin{aligned}
f(s) &= (\tilde{\mu}(\rho_x)f)(s), \\
&= \sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0)(\rho_x(v_0, s_0)f)(s), \\
&= \sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0)e^{-2\pi i \langle x, sv_0 \rangle} f(ss_0), \tag{B.6}
\end{aligned}$$

for all s in S_{n+1} . Let t be an element of S_{n+1} such that $|f(t)| = \max_{s \in S_{n+1}} |f(s)|$.

Then, for all (v_0, s_0) in $\text{Supp}(\mu)$, since f is nonzero function, we have

$$\sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0)e^{-2\pi i \langle x, tv_0 \rangle} \frac{f(ts_0)}{f(t)} = 1. \tag{B.7}$$

This implies, for all (v_0, s_0) in $\text{Supp}(\mu)$, $e^{-2\pi i \langle x, sv_0 \rangle} f(ts_0) = f(t)$ from which we obtain

$$(\rho_x(v_0, s_0)f)(t) = f(t),$$

for all (v_0, s_0) in $\text{Supp}(\mu)$. Since $\text{Supp}(\mu)$ generates A , we find that the restriction function $f|_{S_{n+1}}$ of f in S_{n+1} is a constant function. Hence from equation (B.6), for all s in S_{n+1} we obtain

$$\sum_{(v_0, s_0) \in \text{Supp}(\mu)} \mu(v_0, s_0)e^{-2\pi i \langle x, sv_0 \rangle} = 1,$$

from which we find $e^{-2\pi i \langle x, sv_0 \rangle} = 1$, for all (v_0, s_0) in $\text{Supp}(\mu)$ and s in S . This implies $e^{-2\pi i \langle x, sv_0 \rangle} = 1$ for all v_0 in L and s in S . Hence for some k in \mathbf{Z} , $\langle x, sv_0 \rangle = k$ for all v_0 in L and s in S . Since $sL = L$ for all s in S , we have $\langle x, v \rangle = k$ for some k in \mathbf{Z} , for all v in L . Therefore, x must be an element of L^* , and as an element of E/L^* , $x = 0$.

□

Appendix C

General Random Walk

In Chapter 3 we discussed how to estimate the ultimate behaviour of the symmetric random walk by exploiting the properties of the operator $P(x)$ as given in Section 3.3. We will show here that our technique remains valid and we can arrive at similar conclusion for a general random walk with the transition probability measure μ given by

$$\mu(v, s) = \begin{cases} \mu_0, & \text{if } (v, s) = (r_0, s_0), \\ \mu_i, & \text{if } (v, s) = (0, s_i), i \in \{1, \dots, n\}, \\ 0, & \text{otherwise,} \end{cases} \quad (\text{C.1})$$

where $\sum_{i=0}^n \mu_i = 1$. We assume that the support $\text{Supp}(\mu)$ of μ generates A .

Our generalization of the operator $P(x)$, x in E/L^* is a linear operator $\kappa(x)$ given by

$$(\kappa(x)f)(s) = \mu_0 e^{2\pi i \langle x, sr_0 \rangle} f(ss_0) + \sum_{i=1}^n \mu_i f(ss_i),$$

for all f in \mathbb{C}^S and s in S .

With this generalization, we outline here some our lemmas in 3.3 which remains valid. The technique we employ to evaluate the ultimate behaviour of the walk in Section 3.4 Chapter 3 can be used without difficulties

Lemma C.0.14 *The operator $\kappa(x)$ is self adjoint for each $x \in E/L^*$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined in equation (3.8).*

Proof. A proof of this Lemma can easily be adapted from the proof for Lemma 3.3.1. In fact, a linear combination of self adjoint linear operator with real coefficients is self adjoint.

□

Lemma C.0.15 *Let λ be an eigenvalue of $\kappa(x)$, x in E/L^* . Then the following statements hold.*

1. $|\lambda| \leq 1$.
2. $\kappa(0)$ has 1 as a simple eigenvalue with all nonzero constant functions as the corresponding eigenvectors.
3. If $x \neq 0$, then $(I - \kappa(x))$ is invertible.

Proof. This is proved in Appendix B page 87.

□

Lemma C.0.16 *There is only one eigenvalue of $\kappa(x)$ which tends to one as x tends to zero.*

Proof. The proof given for Lemma 3.3.3 remains valid by changing the operator $P(x)$ with $\kappa(x)$.

□

Lemma C.0.17 *Let $\lambda_0(x)$ be the eigenvalue of $\kappa(x)$ which tends to one as x tends to zero. Then for small x in E/L^* we have*

$$\lambda_0(x) = \lambda_0(-x).$$

Proof. The proof we have given for $P(x)$ is derived from the fact that $\overline{(P(x)\psi)(s)} = (P(-x)\psi)(s)$ for all ψ in \mathbf{C}^S and s in S . But this condition is satisfied by the operator $\kappa(x)$. Therefore Lemma C.0.17 follows. \square

Lemma C.0.18 *Let $\lambda_0(x)$ be the eigenvalue of $\kappa(x)$ which tends to one as x tends to zero. Then we have*

$$\lambda_0(x) = 1 - c\langle x, x \rangle + O(\|x\|^4), \quad (\text{C.2})$$

where c is positive constant given by

$$c = \frac{2\pi\mu}{n}(2\pi + \langle r_0, s_0 b \rangle)$$

and $O(\|x\|^4)$ denotes terms containing fourth and higher powers of x .

Proof. By similar argument for obtaining $\lambda_{0,2} + \alpha_2(x, s)$ and $\beta_1(x, s)$ in page (36), using definition μ in equation (C.1) we obtain

$$\begin{aligned} \lambda_{0,2}(x) + \alpha_2(x, s) &= \mu_0\alpha_2(x, ss_0) - \frac{1}{2}\mu_0(2\pi\langle x, sr_0 \rangle)^2, \\ &\quad - 2\pi\mu_0\langle x, sr_0 \rangle\beta_1(x, ss_0) + \sum_{i=1}^n \mu_i\alpha_2(x, ss_i). \end{aligned} \quad (\text{C.3})$$

and

$$\beta_1(x, s) = \mu_0(\beta_1(x, ss_0) + 2\pi\langle x, sr_0 \rangle) + \sum_{i=1}^n \mu_i\beta_1(x, ss_i). \quad (\text{C.4})$$

In addition, we observe that for all f in \mathbf{C}^S

$$\begin{aligned} \sum_{s \in S} \left(\sum_{i=0}^n \mu_i f(ss_i) \right) &= \sum_{s \in S} (\mu_0 f(ss_0) + \dots + \mu_n f(ss_n)), \\ &= \mu_0 \sum_{s \in S} f(ss_0) + \dots + \mu_n \sum_{s \in S} f(ss_n), \\ &= (\mu_0 + \dots + \mu_n) \sum_{s \in S} f(s), \\ &= \sum_{s \in S} f(s), \end{aligned}$$

since $\sum_{s \in S} f(ss_i) = \sum_{s \in S} f(s)$ for all $i = 0, \dots, n$. With this remark it can be easily seen that equation C.2 is true. To complete the proof, we shall show that c is positive. By similar argument in obtaining equation (3.42) page 39, we have

$$\mu_i \langle r_i, b \rangle = \mu_0 (2\pi - \langle r_0, b \rangle),$$

for all $i = 1, \dots, n$. Let $\mu_M = \min\{\frac{\mu_i}{\mu_0} : i = 1, \dots, n\}$. Then we have

$$\langle r_0, b \rangle < 2\pi - \mu_M \langle r_i, b \rangle,$$

for all $i = 1, \dots, n$. Hence summing both sides over $i = 1, \dots, n$, we obtain

$$\langle r_0, b \rangle < \frac{2\pi n}{n + \mu_M} < 2\pi.$$

Hence we have $\langle r_0, s_0 b \rangle = -\langle r_0, b \rangle > -2\pi$ completing the proof.

□

By similar argument for random walk problem, we conclude that the ultimate behaviour of the general random walk is recurrent for $n = 2$ and transitory for $n > 2$.

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